# KEY

TO THE

# ELEMENTARY COURSE

OF

# PRACTICAL MATHEMATICS,

CONTAINING

DEMONSTRATIONS OF THE RULES, AND FULL OR CONCISE SOLUTIONS OF MOST OF THE EXERCISES IN THE COURSE.

PART I.

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ALGEBRA.



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## PART II. OF KEY.

PRACTICAL GEOMETRY AND MENSURATION.

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# PREFACE.

The Elementary Course, which this volume is intended to accompany, contains merely the *practice* of Mathematics, the *principles* being left to the explanations of the teacher. It was by no means intended that theory should be *excluded:* the Author would most decidedly condemn such a method of teaching any branch of Mathematics; but *to beginners*, as explained in the Preface to the Course, demonstration and explanation come with most effect from the teacher's lips. When rules, exercises, and demonstration are blended together in the same pages, the younger pupil is too apt to regard the whole as a confused mass, and that which is intended to elucidate not unfrequently obscures. On the contrary, when the manual employed contains merely the work to be done and the rules for doing it, while the teacher is left to bring out and establish the principles, either by full and regular demonstration, by shorter remarks made at intervals, or by the Socratic method—of interrogation, the pupil obtains a clear idea of the whole, the teacher is furnished with a convenient class-book, and both teacher and pupil have in their hand that which can be used *readily* as a book of reference.

The method of giving the demonstrations in the form of foot-notes is another alternative; but it is so well known to be unsatisfactory that it is not necessary to dwell upon it.

In *this* volume the teacher is furnished with the necessary material for his explanations. To junior teachers this may be necessary; and even to one more experienced it is

sometimes useful to have at hand a complete view of the business before him, that he may be reminded of what needs demonstration, and of the order in which the arrangement of the text-book requires it to be introduced.

In the demonstrations employed, simplicity has always been aimed at when compatible with perfect accuracy. For this reason particular instead of general expressions are often used to illustrate principles. That method is quite allowable, and is no infringement of strict demonstration, when the particular expressions employed are used only for illustration, and when the reasoning has no peculiar application to those expressions more than to others.*

The *solutions* will seldom be required by the teacher for any difficulty the exercises may present; but it may save him trouble, when operations of some length are necessary, to have the figures ready at his hand for reference. The Author's apology for inserting solutions of other exercises, neither difficult nor laborious, is, that a student, who has not the advantage of a master, may be enabled to instruct himself by the use of the two volumes together, and may find every difficulty removed that can possibly obstruct his progress. The simplest and most detailed solutions will be found at the commencement of the course and of each particular subject, greater conciseness being introduced in subsequent stages.

In the course of this volume the Author has taken the liberty to introduce a new sign, which, he is persuaded, will require no apology as soon as its use is understood, for he is well satisfied that the want of it has been experienced by every Algebraist, and every student of Algebra, in innumerable instances. It is only surprising that the desideratum should not have been supplied long ago. The sign, $=$, is vague, sometimes signifying the mere idea of equality, at other times asserting that such equality exists. At one time it conveys an affirmation, at another time a mere allusion. It is therefore proposed to separate the two uses, giving a distinct sign to each, and *that* with as slight a deviation as possible from established use. The old symbol, $=$, will be retained in the adjective sense, and a new one, $\doteq$, is introduced for the verb. The former is read simply *equal to*, and the latter, *is equal to.*

Suppose, for instance, that the following simple steps

* What else are the diagrams of Euclid?

occurred in an investigation, expressed without the proposed change, thus,—

$$b = 3,\ \text{and}\ c = 5.$$
$$\therefore a^2 = bc = 3 \times 5 = 15.$$

Who can say by which of the three signs in the last line the affirmation is made, or, in other words, on which of them the force of the "$\therefore$" falls? We may make it out by examining for ourselves; but the mere statement, as it stands, is ambiguous, and in more complicated expressions that ambiguity often proves a serious hindrance to the reader. With the alteration now introduced the same statements will be expressed thus,—

$$b \doteq 3,\ \text{and}\ c \doteq 5.$$
$$\therefore a^2 = bc \doteq 3 \times 5 = 15.$$

This is read, "$b$ is equal to 3, and $c$ is equal to 5. Therefore $a^2$, which is equal to $bc$, is equal to 3 times 5, which again is equal to 15."

Another innovation, the Author is convinced, will also prove useful, viz. a distinction between the signs expressing necessary and contingent equality. Thus when we say $a^2 = bc$, the sign $=$ conveys a very different meaning from what it does when we say $3 \times 5 = 15$. In the latter case it expresses identity in actual value, and in the former mere accidental equality under the peculiar values given to $a$, $b$, and $c$. The former might be expressed thus, $\underset{\cdot}{=}$, the latter thus, $=$. With this change, returning to the example above, the last line would stand thus,—

$$\therefore a^2 \underset{\cdot}{=} bc \underset{\cdot}{\doteq} 3 \times 5 = 15,$$

which will be read thus, "Therefore $a^2$, equal, in this instance, to $bc$, is equal, under the present supposition, to 3 times 5, which is identical in value with 15."

Corresponding modifications might be introduced into the signs $>$ and $<$.

Another instance may be given, taken from a standard work, to show the necessity for the innovation proposed. The particular example is not taken as in any way peculiar, but as one among thousands. The equations, in the original, stand thus:—

$$\text{Cos}\ \theta + \cos\phi . \frac{\sin\theta}{\sin\phi} = 0;$$
$$\therefore \sin\phi\cos\theta + \sin\theta\cos\phi = \sin(\phi + \theta) = 0 = \sin\pi.$$

Where, then, does the assertion, in the second line, rest? We can tell only by examining the process. On the system proposed it would be written thus:—

$$\text{Cos } \theta + \cos \phi . \frac{\sin \theta}{\sin \phi} \div 0:$$

$$\therefore \sin \phi \cos \theta + \sin \theta \cos \phi \doteq \sin (\phi + \theta) \div 0 = \sin \pi.$$

These changes, the Author feels confident, would introduce a clearness and precision into the symbolical language of Algebra, of which it has hitherto been destitute. In this volume, however, only the first of the alterations alluded to is brought into use. The others are left, for the present, to the consideration of Mathematicians. Neither has the new sign, $\doteq$, been introduced into the Elementary Course itself, but merely into this volume. In the former it is not necessary, as no demonstrations are introduced and no ambiguity can occur. It is thus left open to the teacher to introduce the use of the new sign or not as he may think proper: where the demonstrations are accompanied by verbal explanations, it is not absolutely necessary: it is when the expressions reach the eye only, and when Algebra is used in reasoning processes, that the Author regards the use of the signs proposed as almost indispensable.

EDINBURGH, *Jan.* 1, 1852.

# KEY

TO

# PRACTICAL MATHEMATICS.

PART FIRST.

# ALGEBRA.

# CONTENTS OF PART I.

# ALGEBRA.

## CHAPTER I.

### DEFINITIONS AND EXPLANATIONS.

*Article* 12.—*Exercise* 1.

**Answer:** 63; 63; 80; 80; 4; 3; 15; 15.

*Exercise* 2.

$$b+c \doteq 8\,; \quad b-c \doteq 4\,; \quad b \pm c \doteq 8 \text{ or } 4\,; \quad a-e \doteq 10\,;$$
$$a+b+c+d+e \doteq 19\,; \quad a-b+c-d \doteq 5\,;$$
$$a \times b \doteq 60\,; \quad b.c \doteq 12\,; \quad abc \doteq 120\,; \quad abcde \doteq 0\,;$$
$$a \div c \doteq 5\,; \quad \frac{b}{c} \doteq 3\,; \quad \frac{d}{c} \doteq \tfrac{1}{2}.$$
$$\frac{a+b}{c} \doteq 8\,; \quad \frac{a-b}{c} \doteq 2\,; \quad \frac{ac-cd}{b} \doteq 3.$$

*Article* 16.—*Exercise.*

**Answer:** 2; 3; and 5.

*Article* 23.—*Exercise.*

**Answer:** 144; 125; 32; 8; 3; and 2.

# CHAPTER II.

## ADDITION.

### Problem I.

*Demonstration of the Rules.*

The rule for the *first Case* is self-evident, if attention is given to the explanation of positive and negative numbers in Chapter I, Article 8. $+$ and $-$, prefixed, merely indicate certain qualities of the quantities to be added, and if all the quantities have the same quality, their sum will also have that quality. As 5 apples added to 3 apples are 8 *apples*, or 5 pears added to 3 pears are 8 *pears*, so $+5$ added to $+3$ will be $+8$, and $-5$ added to $-3$ will be $-8$.

The reason of the rule for *Case* 2 is almost equally simple, for it results, from the same explanation of positive and negative numbers, that $+5$ added to $-5$ is 0, that $+7$ added to $-7$ is 0; and, in general, if $m$ be any number, that $+m$ added to $-m$ is 0. Now, suppose we are required to add $-6$ to $+8$, we know, by Case 1, that $+8 \doteq +6+2$, and if $-6$ be added to this, $-6$ added to $+6$ is 0, and the $+2$ remains as the sum. Again, if $+6$ is to be added to $-8$, we know, from Case 1, that $-8 \doteq -6-2$, and if $+6$ be added to this, $+6$ added to $-6$ is 0, and $-2$ is left as the sum. And, in general, if we are required to add $-n$ to $+m+n$, the result will be $+m$; but if to add $+n$ to $-(m+n)$ or (which is the same thing) to $-m-n$, the result is $-m$.

The rule for *Case* 3 follows from the rules for the two preceding cases. Those for *Cases* 4 and 5 need no explanation.

### Problem II.

None of the rules under this Problem require demonstration.

*Exercise* 9.

$$
\begin{array}{rl}
 & +3(a+b)-5(a-b) \\
 & -6(a+b)+4(a-b) \\
 & +7(a+b)+7(a-b) \\
 & \underline{-1(a+b)+1(a-b)} \\
\text{Ans.} & +3(a+b)+7(a-b).
\end{array}
$$

# CHAPTER III.

## SUBTRACTION.

*Demonstration of the Rule.*

When the quantity to be subtracted forms a part of that from which it is to be taken, we have merely to remove that part, the simplest idea of Subtraction being the removing of a part from the whole. Thus, $+n$ subtracted from $m+n$, gives $m$, and $-q$ subtracted from $p-q$, gives $p$.

But if the subtrahend is no part of the minuend, we may alter the form of the minuend (without changing its value) so that the subtrahend shall be equivalent to a part of it. Thus we know, by the rules of Addition, that $m = m + (n-n) = m+n-n$: and, if we are required to subtract $+n$ from $m$ ($m$ being any quantity, positive or negative), instead of taking $+n$ from $m$ let us take it from its equal, $m+n-n$, and there remains $m-n$; so that subtracting $+n$ is equivalent to adding $-n$. Again, if $-n$ is to be subtracted from $m$, we take it from its equal $m+n-n$, and there remains $m+n$; so that subtracting $-n$ is equivalent to adding $+n$.

The reason of the rule for Subtraction may be rendered still more familiar to beginners by reverting to the illustrations given of positive and negative quantities in Article 8 of Chapter I. Thus, regarding $+$ as indicating property, and $-$, debt, we know that taking away property from any one is equivalent to adding just as much to his debt, or taking away so much of his debt is equivalent to giving him so much property.

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# CHAPTER IV.

## THE VINCULUM.

*Demonstration of the Rules.*

Rules 1 and 2 are mere definitions or explanations.

Rules 3 and 4 are derived respectively from the rules for Addition and Subtraction.

Rule 5 results from 3 and 4.

*Rule* 2.—*Exercise* 1.

Answer: 14; 2; $-14$; and $-2$.

*Exercise* 2.

Ans. 12; 100; 0; 216; 7; 3.

*Rules* 3 *and* 4.—*Exercise* 1.

Ans. $a+b+c$; $a+b-c$; $a-b-c$; $a-b+c$.

*Exercise* 2.

Ans. $10+5+2$; $10+5-2$; $10-5-2$; $10-5+2$.

*Exercise* 3.

Ans. 17; 13; 3; 7.

*Rule* 5.—*Exercise* 1.

Ans. $a+(b+c)$; $a+(b-c)$; $a+(c-b)$.

*Exercise* 2.

Ans. $a-(b+c)$, $a-(b-c)$, $a-(c-b)$.

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## CHAPTER V.

### MULTIPLICATION.

#### Problem I.

*Demonstration of the Rule.*

(1.) $a \times b \doteq ab$, being synonymous expressions. See Ch. I, 11.

(2.) $ab \doteq ba$. Thus 3 times 4 is equal to 4 times 3. This is shown by placing the two factors in rows and cross rows. Thus, in the annexed diagram, we have either 4 rows of 3 in each row, or 3 rows of 4 in each row, according as we take the rows vertically or horizontally. But the *whole number* is the same whether we count

. . . .
. . . .
. . . .

them the one way or the other. That is, $4 \times 3 \doteq 3 \times 4$; or, generally, $ab \doteq ba$.

(3.) $abc$ may either be regarded as $ab \times c$, or as $a \times bc$, since the two expressions are equivalent. That they *are* equivalent is not absolutely self-evident, although familiar use leads us to suppose so. A particular case will elucidate the principle most clearly. We are required to prove that $7 \times 15 \doteq 21 \times 5$, or that to multiply 7 by 15 is equivalent to multiplying 7 by 3, and the product by 5, which may be shown thus. The number $15 \doteq 5$ times 3, whatever the unit be: thus, 15 apples $\doteq$ 5 times (3 apples),* and 15 baskets of apples $\doteq$ 5 times (3 baskets of apples): 15 sevens, then, or 15 times 7, $\doteq$ 5 times (3 sevens) $=$ 5 times $21 = 21 \times 5$.

(4.) $abc$ may also be regarded as $ac \times b$; for $abc \doteq a \times bc = a \times cb$, since $bc \doteq cb$, as shown above; and $a \times cb \doteq ac \times b$, as just proved.

(5.) If, therefore, one quantity is to be multiplied by the product of two others, or the product of two quantities to be multiplied by a third, or if the product of three separate quantities be required, in any of these cases, when the quantities are expressed by letters, we merely write the quantities together, and *that* in any order that may be most convenient. The same rule may evidently be extended, by the same process of reasoning, to any number of quantities; and, if one or more numeral factors occur along with the literal quantities, the product of the former may be taken by itself, and placed as a co-efficient to the product of the latter.

(6.) *The rule for the sign* is less simple, and, as applied to the product of two negative quantities, is not easily perceived by the learner.

That multiplying by a positive number will not alter the sign of the multiplicand is clear, for taking $m$ times any number of things will not alter the character of those things. Thus, $m$ and $n$ being any numbers, $m$ times $n$ pounds will be pounds, $m$ times $n$ miles will be miles; so $m$ times any amount of property will be property, and $m$ times any amount of debt will be debt; $m$ times any height will be height, and $m$ times any depth will be depth. Therefore $m$ times $+n$ will be $+mn$, and $m$ times $-n$ will be $-mn$.

Or thus:—Multiplying by a positive quantity is equi-

* The parenthesis is used here for the algebraical vinculum. Its use makes a nice distinction which must be attended to.

valent to adding the multiplicand to itself as many times as there are units in the multiplier: and, therefore,

$$+5\times 3 = +5+5+5 \doteq +15,$$
$$\text{and} -5\times 3 = -5-5-5 \doteq -15,$$

by the first rule of Addition.

The multiplier, $m$ or 3, is, of course, the same as $+m$ or $+3$.

Having proved that $-$ multiplied by $+$ gives $-$ (or rather quantities with those signs attached to them), it may be supposed that the next case is also proved, viz. that $+$ multiplied by $-$ gives $-$, and, in fact, in many standard works this has been assumed.* But, although $(+n)\times(-m)$ is the same in *result* as $(-n)\times(+m)$, yet in reality the processes are essentially different. We know readily what is meant by $m$ times a negative quantity, such as 7 times a given debt, but it requires explanation to show us what is meant by taking a positive quantity a negative number of times. To ascertain this clearly, let us see what we mean by taking any number so many times. Seven times a gallon is, of course, the same as seven gallons: so seven times a number is the same as seven such numbers added together: 7 times 10, for instance, just means 7 tens. A negative number of times, then, is explained in the same way: (minus 7)† times a gallon is just (minus 7) gallons; and $(-7)$ gallons means 7 gallons subtracted. So $(-7)$ tens signifies 7 tens subtracted; or any number taken $(-7)$ times is identical with 7 times that number subtracted. Thus $10\times(-7)$ will be 70 subtracted, or $-70$; $8\times(-5)$ will be 40 subtracted, or $-40$; and, in general, $(+m)\times(-n)$ will be $mn$ subtracted, or $-mn$.

In like manner, $(-10)\times(-7)$ will be equivalent to 7 times $-10$ subtracted, that is to $-70$ subtracted, which makes $+70$; and, in general, $(-m)\times(-n)$ is the same as $-m\times n$ subtracted, or $+mn$.

In every case, therefore, multiplying any quantity by a negative number changes the sign of that quantity.

We have proved then that $+$ into $+$ gives $+$, that $-$ into $+$ gives $-$, that $+$ into $-$ gives $-$, and that $-$ into $-$ gives $+$; or, as a general rule, that *like signs give* $+$, *and unlike give* $-$.

* It will not do to assume this on the ground of what has already been proved, viz. that $ab$ is $=ba$, for that was proved only on the understanding that both $a$ and $b$ were positive quantities.

† The parenthesis is again used for the algebraical vinculum.

The subject may be still further elucidated to beginners by analogy. Let them be shown by instances, that, when the multiplier and the multiplicand are both positive and the latter constant, the product always diminishes as we diminish the multiplier till both become 0 together; and, consequently, that if we diminish the multiplier still more, the multiplicand may be expected to diminish still more also; so that, when the multiplier becomes negative, the product becomes negative. On the other hand, when the multiplicand is negative and constant, and the multiplier positive, that the product increases as we diminish the multiplier. For instance, $-5\times4 \doteq -20$; but $-5\times3 \doteq -15$, which is greater than $-20$. As the multiplier is continually diminished until it vanishes, the product vanishes at the same time; and consequently, extending the analogy, we may expect that, when the multiplier becomes less than 0, the product will become greater than 0.

The latter method may not be regarded as strict demonstration; but it will show, at least, that there is *nothing strange* in the circumstance that multiplying a negative number by a negative number gives a positive number, but, on the contrary, that it is exactly what we might expect.*

NOTE. The direction given in Note 1 arises, evidently, from the definition of the square of a number; for $a^2 \doteq aa$.

The rule contained in Note 2 is also almost self-evident. For instance, $b^2\times b^3 \doteq b^5$, because $b^2 \doteq bb$ and $b^3 \doteq bbb$; and, consequently, $b^2\times b^3 \doteq bb\times bbb = bbbbb = b^5$.

*Exercises.*

1. Ans. $+cd$, and $+mnxy$.
2. Ans. $-6ab$, and $-49xy$.
3. Ans. $-30x^2y^2$, and $+300abx^2y^2$.
4. Ans. $-28mny^2$.
5. Ans. $+26a^2bcx^2$.
6. Ans. $+72a^{12}$, and $+33a^2bcx^5$.
7. Ans. $-34m^4n^5$.

* The method, adopted by many writers of authority, of demonstrating the principle from binomial factors, is very unsatisfactory, and is contrary to our usual approved practice of establishing the complex by means of the simple, and not the simple by means of the complex.

## PROBLEM II.

*Demonstration of the Rule.*

The rule rests on the fact that the product of the sum of two quantities by another quantity is equal to the sum of their separate products by that quantity; and that the product of the difference of two quantities by another quantity is equal to the difference of their separate products. Thus $(m+n) \times a \doteq am + an$, and $(m-n) \times a \doteq am - an$. The truth of that we may prove by placing $(m+n)$ or $(m-n)$ objects in a row and taking $a$ such rows. Thus, if $m$ is 3; $n$, 2; and $a$, 4; our row, for the sum, will contain $(3+2)$ or 5, and we shall have 4 rows of 5 in each row, which we see at once is equal to 4 rows of 2 in each added to 4 of 3 in each. Therefore $(3+2) \times 4 \doteq (3 \times 4) + (2 \times 4)$.

So, for the difference, our row will contain $3-2$, or 1, and we shall have 4 rows of 1 in each, which is identical with 4 rows of 3, diminished by 4 rows of 2. Therefore $(3-2) \times 4 \doteq (3 \times 4) - (2 \times 4)$.

If $(m+n) \times a \doteq am + an$, and if $(m-n) \times a \doteq am - an$, it follows that $(m+n+p) \times a \doteq am + an + ap$, and $(m-n+p) \times a \doteq am - an + pn$; and, in the same manner, that $(m+n-p) \times a \doteq ma + na - pa$, and $(m-n-p) \times a \doteq ma - na - pa$; and, generally, that the same rule is applicable to any number of quantities connected in any way by the signs $+$ and $-$.

If the multiplier is the compound quantity and the multiplicand the simple quantity, the principle is the same.

## PROBLEM III.

The demonstration of the rule for this Problem is the same as that for the preceding. We take the multiplicand as *one* quantity $(a)$, and we are required to show that $a \times (m \pm n) \doteq am \pm an$, which has been demonstrated under Problem II. If there are more than two terms in the *multiplier, the* principle is the same.

NOTE. The processes in this and the preceding Problem, when all the quantities are positive, correspond exactly to those with which we are familiar in common Arithmetic. When we multiply 43 by 2, we proceed as in Problem II; for $43 \times 2 \doteq (40 \times 2) + (3 \times 2)$; and when we multiply 39 by 25 we proceed as in Problem III; for $39 \times 25 \doteq (39 \times 20) + (39 \times 5)$.

*Exercise* 6.

$$\begin{array}{rllll}
 & 9a^2 + & 6ab + & 4b^2 & \\
 & 3a - & 2b & & \\
\hline
 & 27a^3 + & 18a^2b + & 12ab^2 & \\
 & & -18a^2b - & 12ab^2 - & 8b^3 \\
\hline
\text{Ans.} & 27a^3 & * & * & -8b^3. \\
\hline\hline
\end{array}$$

*Exercise* 7.

$$\begin{array}{rlllll}
 & -4m^2 + & 10mn - & 25n^2 & & \\
 & -4m^2 - & 10mn - & 25n^2 & & \\
\hline
 & +16m^4 - & 40m^3n + & 100m^2n^2 & & \\
 & & +40m^3n - & 100m^2n^2 + & 250mn^3 & \\
 & & & +100m^2n^2 - & 250mn^3 + & 625n^4. \\
\hline
\text{Ans.} & 16m^4 & * & +100m^2n^2 & & +625n^4. \\
\hline\hline
\end{array}$$

*Exercise* 8.

$$\begin{array}{rlllllll}
 & a^3 - & 3a^2x + & 3ax^2 - & x^3 & & & \\
 & a^3 + & 3a^2x + & 3ax^2 + & x^3 & & & \\
\hline
 & a^6 - & 3a^5x + & 3a^4x^2 - & 1a^3x^3 & & & \\
 & & +3a^5x - & 9a^4x^2 + & 9a^3x^3 - & 3a^2x^4 & & \\
 & & & +3a^4x^2 - & 9a^3x^3 + & 9a^2x^4 - & 3ax^5 & \\
 & & & & +1a^3x^3 - & 3a^2x^4 + & 3ax^5 - & x^6 \\
\hline
\text{Ans.} & a^6 & * & -3a^4x^2 & * & +3a^2x^4 & * & -x^6. \\
\hline\hline
\end{array}$$

*Exercise* 10.

$$
\begin{array}{r}
8a^3+10a^2+12a\ +14 \\
4a^3-\ 5a^2+\ 6a\ -\ 7 \\
\hline
32a^6+40a^5+48a^4+56a^3 \qquad\qquad\qquad\qquad\qquad \\
-40a^5-50a^4-60a^3-70a^2 \qquad\qquad\qquad \\
+48a^4+60a^3+72a^2+84a \qquad\quad \\
-56a^3-70a^2-84a-98 \\
\hline
\text{Ans. } 32a^6\ *\ +46a^4\ *\ -68a^2\ *\ -98.
\end{array}
$$

## Problem IV.

The rule requires no *demonstration*; it merely directs that to be done which is necessarily implied in the terms of the problem.

*Exercise* 4.

$$(a+b)\times(a-b) \doteq a^2-b^2.$$
$$(a^2+b^2)\times(a^2-b^2) \doteq a^4-b^4.$$

*Exercise* 5.

$$(a+b-c)\times(a-b+c) \doteq a^2-b^2+2bc-c^2.$$

*Exercise* 6.

$$(x-3)\times(x+3) \doteq x^2-9.$$
$$(x-5)\times(x+5) \doteq x^2-25.$$
$$(x^2-9)\times(x^2-25) = x^4-34x^2+225.$$

*Exercise* 7.

$$(x^2+xy+y^2)\times(x-y) \doteq x^3-y^3.$$
$$(x^2-xy+y^2)\times(x+y) \doteq x^3+y^3.$$
$$(x^3-y^3)\times(x^3+y^3) \doteq x^6-y^6.$$

# CHAPTER VI.

## DIVISION.

### *Demonstration of the Rule.*

Division is the reverse of Multiplication. In Multiplication we are required to take a certain quantity (or number) a given number of times. In Division we are required to ascertain how many times a given number is taken to make another given number. That other number, then, (the dividend) we suppose to be the product of the other two, one of which (the divisor) is given, the other (the quotient) being to be found. But if the dividend be the product of the divisor and quotient, it must be made up of the factors of the divisor and quotient together. Consequently, in the case of simple quantities, if we desire to find the quotient, we have merely to separate the factors of the dividend, and take away from them those of the divisor, when those remaining will be the factors of the quotient. For instance, if we are required to divide $abcd$ by $ab$, what we mean is, to find by what number $ab$ must be multiplied to make $abcd$: now, whatever that number be, we know from multiplication that, when it is found, its factors along with those of $ab$ must just make up $abcd$, consequently the number must be $cd$, or, in other words, must be the product of all the other factors of $abcd$.

This applies to the case in which all the factors of the divisor are found explicitly in the dividend. If they are not, another process is required. Let us take, first, the case of $a \div b$, in which $b$ has no factor contained in $a$. In that case the quotient is the fraction $\frac{a}{b}$; for we shall afterwards prove, when we come to the head of Fractions, that $\frac{a}{b} \times b \doteq a$.

Next, to divide $ac$ by $bc$, $a$ and $b$ being understood to have no common factor. Since $ac \doteq \frac{a}{b} \times bc$, $ac \div bc \doteq \left(\frac{a}{b} \times bc\right) \div bc = \frac{a}{b}$. Or thus:—since we have proved that $a \div b \doteq \frac{a}{b}$, $a$ and $b$ being any numbers whatever, it fol-

lows that $ac \div bc \doteq \frac{ac}{bc}$, and this fraction reduced to its lowest terms, is $\frac{a}{b}$. By either way the common factor has disappeared.

In the preceding demonstration, however, the subject of Fractions has been anticipated. The proof given to learners must therefore be incomplete in the present stage of their progress, and the rule must be assumed without demonstration till they have attained a more advanced position.

The *rule for the sign* follows evidently from that of Multiplication.

*Exercise* 1.

Answers: $3a$; $\frac{5x}{4z}$; $-2n$; and $+\frac{7c}{3}$.

*Exercise* 2.

Ans. $a^3$; $-12\frac{c^2}{b^2}$; $-\frac{1}{8xy^2z^3}$; and $+\frac{6x^3y}{5}$.

*Exercise* 3.

Ans. $+\frac{7}{6}ac$; and $-\frac{9}{11}xy^2$

*Exercise* 4.

Ans. $-3\frac{b}{a}$; and $+13\frac{my^3}{n}$.

## Problem II.

The demonstration of the rule rests on the principle that $(a \pm b) \div m \doteq \frac{a}{m} \pm \frac{b}{m}$; and *that* we know, because $\left(\frac{a}{m} \pm \frac{b}{m}\right) \times m \doteq a \pm b$.

## Problem III.

The rule depends upon the same principle as that of *Problem* II, viz. that the whole of any quantity divided by

another quantity will be equal to the sum of the quotients arising from dividing the several parts of the former by the latter; for the dividend is made up of the several products which make their appearance in the course of the operation, and of the remainder, added together. Thus, in the Example, the dividend, $a^3-6a^2b+12ab-8b^3$, is equivalent to

$$\left\{\begin{array}{lll} a^3-2a^2b & & \\ & -4a^2b+8ab^2 & \\ & & +4ab^2-8b^3. \end{array}\right\}$$

Now the first line of this, divided by $a-2b$, gives $a^2$; the second gives $-4ab$; and the third, $+4b^2$. Therefore the whole quantity divided by $a-2b$ is $a^2-4ab+4b^2$. If there had been a remainder, it would have formed another part of the dividend, and, as it would not have been properly divisible by $a-2b$, it would have been attached to the quotient in a fractional form, for we have before demonstrated that $p \div q \doteq \frac{p}{q}$, $p$ and $q$ being any quantities whatever.

*Exercise* 5.

$$x^2-2xy+y^2)\quad (x^3-3x^2y+3xy^2-y^3.$$

$$\begin{array}{l} x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5 \\ \underline{x^5-2x^4y+\phantom{0}1x^3y^2} \\ \quad -3x^4y+\phantom{0}9x^3y^2-10x^2y^3 \\ \quad \underline{-3x^4y+\phantom{0}6x^3y^2-\phantom{0}3x^2y^3} \\ \qquad\qquad +\phantom{0}3x^3y^2-\phantom{0}7x^2y^3+5xy^4 \\ \qquad\qquad \underline{+\phantom{0}3x^3y^2-\phantom{0}6x^2y^3+3xy^4} \\ \qquad\qquad\qquad\qquad -\phantom{0}1x^2y^3+2xy^4-y^5 \\ \qquad\qquad\qquad\qquad \underline{\underline{-\phantom{0}1x^2y^3+2xy^4-y^5}} \end{array}$$

*Exercise* 6.

$$2x^2+2xy)-8x^4-8x^3y+6x^2y^2+6xy^3(-4x^2+3y^2.$$

$$\begin{array}{l} \underline{-8x^4-8x^3y} \\ \qquad\qquad +6x^2y^2+6xy^3 \\ \qquad\qquad \underline{\underline{+6x^2y^2+6xy^3.}} \end{array}$$

### *Exercise* 7.

$$
\begin{array}{l}
3a-6)12a^4 \ldots\ldots\ldots\ldots\ldots\ldots -192(4a^3+8a^2+16a+32. \\
\quad 12a^4-24a^3 \\
\hline
\quad\quad +24a^3 \\
\quad\quad +24a^3-48a^2 \\
\hline
\quad\quad\quad +48a^2 \\
\quad\quad\quad +48a^2-96a \\
\hline
\quad\quad\quad\quad +96a-192 \\
\quad\quad\quad\quad +96a-192. \\
\hline\hline
\end{array}
$$

### *Exercise* 8.

$$
\begin{array}{l}
4z^2-9)64z^6 \ldots\ldots\ldots\ldots\ldots -729(16z^4+36z^2+81. \\
\quad 64z^6-144z^4 \\
\hline
\quad\quad +144z^4 \\
\quad\quad +144z^4-324z^2 \\
\hline
\quad\quad\quad +324z^2-729 \\
\quad\quad\quad +324z^2-729. \\
\hline\hline
\end{array}
$$

### *Exercise* 9.

$$
\begin{array}{l}
2a^3-3a^2b+4ab^2-5b^3) \qquad\qquad (5a^2-3ab-b^2. \\
\quad 10a^5-21a^4b+27a^3b^2-34a^2b^3+11ab^4+5b^5. \\
\quad 10a^5-15a^4b+20a^3b^2-25a^2b^3 \\
\hline
\quad\quad -\ 6a^4b+\ 7a^3b^2-\ 9a^2b^3+11ab^4 \\
\quad\quad -\ 6a^4b+\ 9a^3b^2-12a^2b^3+15ab^4 \\
\hline
\quad\quad\quad -\ 2a^3b^2+\ 3a^2b^3-\ 4ab^4+5b^5 \\
\quad\quad\quad -\ 2a^3b^2+\ 3a^2b^3-\ 4ab^4+5b^5. \\
\hline\hline
\end{array}
$$

### *Exercise* 13.

$$
\begin{array}{l}
a^2-2ab+b^2)a^3-3a^2b+2ab^2-2b^3(a-b-\dfrac{+ab^2+b^3}{a^2-2ab+b^2}. \\
\quad a^3-2a^2b+1ab^2 \\
\hline
\quad\quad -1a^2b+1ab^2-2b^3 \\
\quad\quad -1a^2b+2ab^2-1b^3 \\
\hline
\quad\quad\quad -\ ab^2-1b^3. \\
\hline\hline
\end{array}
$$

### *Exercise* 15.

$$
\begin{array}{l}
8a^3-36a^2b+54ab^2-27b^3) \qquad (4a^2-12ab+9b^2. \\
\quad 32a^5-240a^4b+720a^3b^2-1100a^2b^3+800ab^4-240b^5 \\
\quad \underline{32a^5-144a^4b+216a^3b^2-\ \ 108a^2b^3} \\
\qquad\quad -\ 96a^4b+504a^3b^2-\ \ 992a^2b^3+800ab^4 \\
\qquad\quad \underline{-\ 96a^4b+432a^3b^2-\ \ 648a^2b^3+324ab^4} \\
\qquad\qquad\qquad +\ \ 72a^3b^2-\ \ 344a^2b^3+476ab^4-240b^5 \\
\qquad\qquad\qquad \underline{+\ \ 72a^3b^2-\ \ 324a^2b^3+486ab^4-243b^5} \\
\qquad\qquad \text{Remainder, } \underline{\underline{-\quad 20a^2b^3-\ \ 10ab^4+\quad 3b^5.}}
\end{array}
$$

### *Note* 4.—*Exercise* 10.

$$
\begin{array}{l}
1-a)1+a(1+2a+2a^2+2a^3+\&c. \\
\qquad \underline{1-a} \\
\qquad\ +2a \\
\qquad\ \underline{+2a-2a^2} \\
\qquad\qquad +2a^2 \\
\qquad\qquad \underline{+2a^2-2a^3} \\
\qquad\qquad\qquad \underline{\underline{+2a^3.}}
\end{array}
$$

### *Exercise* 12.

$$
\begin{array}{l}
1+x)1 \qquad (1-x+x^2-\&c. \\
\qquad \underline{1+x} \\
\qquad\ -x \\
\qquad\ \underline{-x-x^2} \\
\qquad\qquad \underline{\underline{+x^2.}}
\end{array}
$$

### *Exercise* 15.

$$
\begin{array}{l}
1-2a+a^2)1 \qquad\qquad (1+2a+3a^2+4a^3+\&c. \\
\qquad\qquad \underline{1-2a+1a^2} \\
\qquad\qquad\ +2a-1a^2 \\
\qquad\qquad\ \underline{+2a-4a^2+2a^3} \\
\qquad\qquad\qquad +3a^2-2a^3 \\
\qquad\qquad\qquad \underline{+3a^2-6a^3+3a^4} \\
\qquad\qquad\qquad\qquad \underline{\underline{+4a^3-3a^4.}}
\end{array}
$$

*Exercise* 7.

$$3a-6)12a^4 \ldots\ldots\ldots\ldots\ldots\ldots -192(4a^3+8a^2+16a+32.$$
$$12a^4-24a^3$$
$$+24a^3$$
$$+24a^3-48a^2$$
$$+48a^2$$
$$+48a^2-96a$$
$$+96a-192$$
$$+96a-192.$$

*Exercise* 8.

$$4z^2-9)64z^6 \ldots\ldots\ldots\ldots\ldots\ldots -729(16z^4+36z^2+81.$$
$$64z^6-144z^4$$
$$+144z^4$$
$$+144z^4-324z^2$$
$$+324z^2-729$$
$$+324z^2-729.$$

*Exercise* 9.

$$2a^3-3a^2b+4ab^2-5b^3) \qquad (5a^2-3ab-b^2.$$
$$10a^5-21a^4b+27a^3b^2-34a^2b^3+11ab^4+5b^5.$$
$$10a^5-15a^4b+20a^3b^2-25a^2b^3$$
$$-\ 6a^4b+\ 7a^3b^2-\ 9a^2b^3+11ab^4$$
$$-\ 6a^4b+\ 9a^3b^2-12a^2b^3+15ab^4$$
$$-\ 2a^3b^2+\ 3a^2b^3-\ 4ab^4+5b^5$$
$$-\ 2a^3b^2+\ 3a^2b^3-\ 4ab^4+5b^5.$$

*Exercise* 13.

$$a^2-2ab+b^2)a^3-3a^2b+2ab^2-2b^3(a-b-\frac{+ab^2+b^3}{a^2-2ab+b^2}.$$
$$a^3-2a^2b+1ab^2$$
$$-1a^2b+1ab^2-2b^3$$
$$-1a^2b+2ab^2-1b^3$$
$$-\ ab^2-1b^3.$$

*Exercise* 15.

$$
\begin{array}{l}
8a^3-36a^2b+54ab^2-27b^3) \qquad\qquad (4a^2-12ab+9b^2. \\
\quad 32a^5-240a^4b+720a^3b^2-1100a^2b^3+800ab^4-240b^5 \\
\quad 32a^5-144a^4b+216a^3b^2-\ \ 108a^2b^3 \\
\hline
\qquad\quad -\ \ 96a^4b+504a^3b^2-\ \ 992a^2b^3+800ab^4 \\
\qquad\quad -\ \ 96a^4b+432a^3b^2-\ \ 648a^2b^3+324ab^4 \\
\hline
\qquad\qquad\qquad\quad +\ \ 72a^3b^2-\ \ 344a^2b^3+476ab^4-240b^5 \\
\qquad\qquad\qquad\quad +\ \ 72a^3b^2-\ \ 324a^2b^3+486ab^4-243b^5 \\
\hline
\qquad\qquad\text{Remainder,}\ -\ \ 20a^2b^3-\ \ 10ab^4+\ \ 3b^5.
\end{array}
$$

*Note* 4.—*Exercise* 10.

$$
\begin{array}{l}
1-a)1+a(1+2a+2a^2+2a^3+\&c. \\
\qquad 1-a \\
\hline
\qquad\ +2a \\
\qquad\ +2a-2a^2 \\
\hline
\qquad\qquad\quad +2a^2 \\
\qquad\qquad\quad +2a^2-2a^3 \\
\hline
\qquad\qquad\qquad\quad +2a^3.
\end{array}
$$

*Exercise* 12.

$$
\begin{array}{l}
1+x)1 \qquad (1-x+x^2-\&c. \\
\qquad 1+x \\
\hline
\qquad\ -x \\
\qquad\ -x-x^2 \\
\hline
\qquad\qquad +x^2.
\end{array}
$$

*Exercise* 15.

$$
\begin{array}{l}
1-2a+a^2)1 \qquad\qquad (1+2a+3a^2+4a^3+\&c. \\
\qquad\qquad 1-2a+1a^2 \\
\hline
\qquad\qquad +2a-1a^2 \\
\qquad\qquad +2a-4a^2+2a^3 \\
\hline
\qquad\qquad\qquad\quad +3a^2-2a^3 \\
\qquad\qquad\qquad\quad +3a^2-6a^3+3a^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad +4a^3-3a^4.
\end{array}
$$

# CHAPTER VII.

## COMMON MEASURES.

*Definitions.—Exercise* 1.

Ans. 1, 2, 3, 4, 6, and 12.

*Exercise* 2.

Ans. 1, 2, 3, 5, 6, 10, 15, and 30.

*Exercise* 3.

Ans. 1, 2, 3, and 6.

*Exercise* 4.

Ans. 1, 2, 4, $a$, $b$, $2a$, $2b$, $4a$, $4b$, $ab$, $2ab$, and $4ab$.

*Exercise* 5.

Ans. 1, 2, 3, 6, $a$, $c$, $2a$, $2c$, $3a$, $3c$, $6a$, $6c$, $ac$, $2ac$, $3ac$, and $6ac$.

*Exercise* 6.

Ans. 1, 2, $a$, and $2a$.

### PROBLEM.

*Demonstration of the Rule.*

Let $A$ be the product of all the factors common to both quantities, and let $B$ and $C$ be the products of the remaining factors in each quantity. Then the two quantities are $A.B$ and $A.C$. Now $A$ evidently measures both, and no quantity greater than $A$ can measure both; for, if any number measure $A.B$, it must either be one of the factors of $A$ or of $B$, or the product of two or more of them, and, in like manner, any number that measures $A.C$

must either be one of the factors of $A$ or of $C$, or the product of two or more of them. Every common measure, therefore, must be made up of factors common to both $A.B$ and $A.C$. But there are none such except $A$ or its factors. Therefore $A$ is the greatest common measure of A.B and A.C.

We here assume that a number is the product of all its prime factors, or, in other words, that if all the prime numbers which severally measure another number be multiplied together, they will just make that number; and, consequently, that a number cannot be the product of two different sets of prime factors. This is seldom proved in elementary treatises, being regarded as almost self-evident, or much oftener overlooked altogether. The demonstration may be too abstruse for beginners, but it is necessary to give it here to complete the theory of common measures. The Teacher may either introduce it to his pupils or not, according to his own discretion, but probably it will be better at this stage, merely to state the fact, leaving its demonstration to a more advanced stage.

If possible let $am$ be $= bn$, $a$, $b$, $m$, and $n$ being prime numbers, or factors prime to each other. Then the g. c. m. of $a$ and $n$ will be 1; but, if we attempt to find it by the common arithmetical process, which will be demonstrated under Case 2, we divide $a$ by $n$, the quotient being $q$, and the remainder $r$, continuing the operation till there is no remainder.

$$\begin{array}{l} n)\ a(q \\ \quad\ nq \\ \quad\ \overline{\quad} \\ \quad\ \ r, \&c. \end{array}$$

The quotient $q$ is always a whole number, because, as usual in dividing, the nearest whole number is taken for the quotient, and, since $q$ and $n$ are both integers, $nq$ must be an integer; and $a$ and $nq$ being integers, their difference $r$ must also be an integer. Every quotient, then, throughout the process, is an integer, and therefore every remainder and every divisor will also be integers. Consequently the last remainder cannot be less than 1; and it cannot be greater, for then $a$ and $n$ would have a common measure greater than unity, which is inconsistent with the hypothesis of their being prime to each other.

Now if the first dividend and divisor were both doubled, the quotient, $q$, would remain the same, but the product $nq$ and the remainder $r$, would be doubled; and, consequently, throughout the operation, every divisor, divi-

dend, and remainder would be doubled, but the quotients would remain all unchanged and all integers.

A corresponding result would follow if the first divisor and dividend were both trebled, or both multiplied by any number, whole or fractional.

If, then, the first divisor and dividend were each multiplied by the fraction $\frac{m}{n}$, the dividend becoming $\frac{m}{n}a$ and the divisor $m$, the process would stand thus,—

$$\begin{array}{l} m)\frac{m}{n}a(q \\ \phantom{m)}mq \\ \phantom{m)}\overline{\phantom{mq}} \\ \phantom{m)}\frac{m}{n}r, \text{ \&c.} \\ \phantom{m)}\overline{\overline{\phantom{mq\ \&c.}}} \end{array}$$

When we should come to the last remainder, which, as the process stood at first, was 1, it would now be $\frac{m}{n}$. Now, since $ma \doteq nb$, $\frac{m}{n}a \doteq b$; and, consequently, is an integer. But, since $m$ and $q$ are both integers, $mq$ is an integer. Therefore the remainder $\frac{m}{n}r$ is also an integer.

In the same way we prove that every remainder is an integer till we come to the last, viz. $\frac{m}{n}$: but if $\frac{m}{n}$ is an integer, $n$ measures $m$, and $m$ and $n$ cannot be prime numbers or factors prime to each other.

Hence we prove that $am$ cannot be $= bnp$, $a$, $m$, $b$, $n$, and $p$, being prime factors, by taking $bn$ as one factor and $p$ as the other; for it has just been shown that if $b$ and $n$ are prime to $a$ and $m$, $bn$ will also be prime to each of them: and in the same manner we demonstrate that the same property holds good whatever be the number of factors.

## Case 1.

No additional demonstration is required in this case. We have already proved that the greatest common measure of the two quantities is the product of all the factors common to both; the common *letters* are seen by inspection, and the truth of the arithmetical process for finding the greatest common measure of the co-efficients will be proved under Case 2.

### *Exercises.*

*Answers.* (1), $ac$; (2), $a^2x$; (3), $3amn$; (4), $14ab^2c^2d$.

## CASE 2.

The demonstration of the particular rule in this case requires to be given at some length. We commence with two almost self-evident theorems.

THEOREM I. *If a quantity measure two others, it will measure their sum and also their difference.* Thus, if 5 is contained in 30 six times, and in 20 four times, it is contained in 50 $(6+4)$ times $=10$ times, because $5\times(6+4)\doteq(5\times 6)+(5\times 4)$; and since 6 and 4 are both whole numbers, $6+4$, or 10, must also be a whole number: therefore 5, being contained in 50 an integral number of times, measures 50. In like manner 5 measures $30-20$, or 10, being contained in it $(6-4)$ times.

THEOREM II. *If a number measure another, it will measure any multiple of it.* This arises obviously from the preceding theorem, since multiplication is equivalent to successive additions of the same quantity.

To apply these theorems to the demonstration of the rule for Case 2, we may take an arithmetical example, the reasoning employed upon which, it will be observed, will hold good in every similar case, and will be equally applicable to quantities algebraically expressed, because there is nothing in the nature of the proof peculiar to the particular numbers employed for illustration.

Let it be required to find the greatest common measure of 114 and 33, which may properly be taken as instances of compound quantities, since $33\doteq 30+3=(3\times 10)+3$, and $114\doteq 100+10+4=(1\times 10^2)+(1\times 10)+4$. Proceeding by the rule, the work will stand as in the margin, 3 being found for the g. c. m.*

```
33)114(3
    99
    —
    15)33(2
       30
       —
        3)15(5
          15
          ==
```

We have, then, two points to make good:—*first*, that 3 must measure both 33 and 114, and *second*, that no number greater than 3 can possibly do so.

To establish the first point, we commence at the end of the operation and trace the steps backward. Since there is no remainder at the close of the process, 3 is contained just 5

* For brevity we shall express, by these letters, the greatest common measure.

times in 15 : therefore 3 measures 15 ; and, if it measure 15 it will measure twice 15, or 30 (Theorem II). It also measures itself; and if it measure 30 and 3, it must (by Th. I) measure $30 + 3$, or 33. If it measure 33 it will measure 3 times 33, or 99. Measuring 99 and 15 it must measure $99 + 15$, or 114. It therefore measures both 33 and 114, and *must* do so.

But, *secondly*, can no number greater than 3 measure 33 and 114? If it can, it must measure 3 times 33 or 99 (Th. I), and next $114 - 99$, or 15 (Th. II): if it measure 15 it will measure 30; and if it measure both 33 and 30 it will measure $33 - 30$, or 3: but no number greater than 3 can measure 3. Therefore no number greater than 3 can measure both 33 and 114.

Consequently 3 is the g. c. m. of 33 and 114.

*Exercise* 1.

$$
\begin{array}{l}
a^2 - b^2)a^3 + a^2b - ab^2 - b^3(a + b. \\
\qquad\quad a^3 \qquad\quad - ab^2 \\
\qquad\quad\quad + a^2b \qquad\quad - b^3 \\
\qquad\quad\quad + a^2b \qquad\quad - b^3
\end{array}
$$

*Exercise* 2.

$$
\begin{array}{l}
a^5 - 2a^3 + 5a^2 - 10)3a^6 - 6a^4 + 16a^3 - 30a + 5(3a. \\
\qquad\qquad\qquad\qquad 3a^6 - 6a^4 + 15a^3 - 30a \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad a^3 \qquad + 5
\end{array}
$$

$$
\begin{array}{l}
a^3 + 5)a^5 - 2a^3 + 5a^2 - 10(a^2 - 2. \\
\qquad\quad a^5 \qquad\quad + 5a^2 \\
\qquad\quad - 2a^3 \qquad\quad - 10 \\
\qquad\quad - 2a^3 \qquad\quad - 10.
\end{array}
$$

*Exercise* 3.

$$
\begin{array}{l}
x^4 - 3x^2 + 2)x^6 - 3x^4 + 3x^2 - 2(x^2 \\
\qquad\qquad\quad x^6 - 3x^4 + 2x^2 \\
\qquad\qquad\qquad\qquad x^2 - 2)x^4 - 3x^2 + 2(x^2 - 1 \\
\qquad\qquad\qquad\qquad\qquad\quad x^4 - 2x^2 \\
\qquad\qquad\qquad\qquad\qquad\quad - 1x^2 + 2 \\
\qquad\qquad\qquad\qquad\qquad\quad - 1x^2 + 2.
\end{array}
$$

NOTE 1. The reason of the permission given in Note 1 is, that if $+a$ be a measure of any quantity, $b$, then $-a$ must also measure $b$; for if $b \div (+a) \doteq c$, then $b \div (-a \doteq -c$.

NOTE 2. The direction given in Note 2 rests on the principle that, if $abc$ and $ade$ be two quantities having $a$ for their g. c. m., and consequently the two quantities $bc$ and $de$ incommensurable (that is, without any common factor), then the g. c. m. will not be changed if we remove, from either quantity, any of the factors which have no common measure. Thus $a$, being the g. c. m. of $abc$ and $ade$, is also the g. c. m. of $ab$ and $ade$, or of $abc$ and $ad$, since the g. c. m. of two quantities is the product of all the factors common to both. In like manner the g. c. m. of two quantities will not be altered if we introduce a new factor into either, provided that factor has no c. m. with any of the factors of the other quantity. Thus the g. c. m. of $abc$ and $adef$ will also be $a$, if $f$ contain no factor found in $b$ or $c$.

Now since, in this Case, our given quantities consist of compound factors only, no simple factor can measure either of them. We may therefore, without altering the g. c. m., multiply either of them by any simple factor, or divide any remainder, in which a simple factor makes its appearance, by that factor, or multiply any dividend by any simple factor which is not also a factor of the corresponding divisor, or commensurable with any simple factor contained in it.

*Exercise* 4.

$$6a^2 + 7ax - 3x^2)6a^2 + 11ax + 3x^2(1$$
$$6a^2 + 7ax - 3x^2$$
$$\overline{2x)4ax + 6x^2}$$
$$2a + 3x\;)6a^2 + 7ax - 3x^2(3a - x$$
$$6a^2 + 9ax$$
$$\overline{-2ax - 3x^2}$$
$$-2ax - 3x^2.$$

*Exercise* 5.

$$
\begin{array}{r|l}
8x^3-4x^2-2x+1) & 12x^3+\ \ 4x^2-3x-1 \\
 & \qquad\qquad\qquad\quad 2 \\ \hline
 & 24x^3+\ \ 8x^2-6x-2(3 \\
 & 24x^3-12x^2-6x+3 \\ \hline
 & \quad 5)+20x^2 \qquad -5 \\ \hline
 & \qquad\ \ 4x^2 \qquad -1.
\end{array}
$$

$4x^2-1$ will be found to divide $8x^3-4x^2-2x+1$ without leaving a remainder, and is therefore the g. c. m.

*Exercise* 6.

$$
\begin{array}{l}
x^2+3x-4)x^3-3x^2-10x(x \\
\qquad\qquad\quad x^3+3x^2-\ \ 4x \\ \hline
\qquad\quad -6x)-6x^2-\ \ 6x \\ \hline
\qquad\qquad\qquad\quad x+\ 1)x^2+3x-4(x+2 \\
\qquad\qquad\qquad\qquad\qquad\ x^2+\ \ x \\ \hline
\qquad\qquad\qquad\qquad\qquad\qquad 2x-4 \\
\qquad\qquad\qquad\qquad\qquad\qquad 2x+2 \\ \hline
\qquad\qquad\qquad\qquad\qquad\quad -6)-6 \\ \hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad +1
\end{array}
$$

*Exercise* 7.

$$
\begin{array}{l}
-x^2+4a^2)4x^3-8ax^2-\ \ 1a^2x+\ \ 2a^3(-4x+8a. \\
\qquad\qquad\quad 4x^3 \qquad\quad -16a^2x \\ \hline
\qquad\qquad\qquad -8ax^2+15a^2x+\ \ 2a^3 \\
\qquad\qquad\qquad -8ax^2 \qquad\quad +32a^3 \\ \hline
\qquad\qquad\qquad\qquad 15a^2)15a^2x-30a^3 \\ \hline
\qquad\qquad\qquad\qquad\qquad\quad x-\ \ 2a)-x^2+4a^2(-x-2a. \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ -x^2+2ax \\ \hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2ax+4a^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2ax+4a^2.
\end{array}
$$

The answer is either $x-2a$ or $2a-x$, by Note 1. Or we might have begun, if we had chosen, by changing the *signs of* the first divisor.

*Exercise* 8.

$$
\begin{array}{l}
a^3 - a^2x - ax^2 + x^3) \qquad\qquad\qquad\qquad (a^2 - 4ax + 7x^2. \\
\quad a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + \ \ 5ax^4 - x^5 \\
\quad \underline{a^5 - 1a^4x - \ \ 1a^3x^2 + \ \ 1a^2x^3} \\
\qquad -4a^4x + 11a^3x^2 - 11a^2x^3 + \ \ 5ax^4 \\
\qquad \underline{-4a^4x + \ \ 4a^3x^2 + \ \ 4a^2x^3 - \ \ 4ax^4} \\
\qquad\qquad + \ \ 7a^3x^2 - 15a^2x^3 + \ \ 9ax^4 - 1x^5 \\
\qquad\qquad \underline{+ \ \ 7a^3x^2 - \ \ 7a^2x^3 - \ \ 7ax^4 + 7x^5} \\
\qquad\qquad\qquad -8x^3) - \ \ 8a^2x^3 + 16ax^4 - 8x^5 \\
\qquad\qquad\qquad\qquad + \ \ a^2 \ \ - \ \ 2ax \ + \ x^2
\end{array}
$$

$$
\begin{array}{l}
a^2 - 2ax + x^2)a^3 - \ \ a^2x - \ \ ax^2 + x^3(a + x \\
\qquad\qquad\quad \underline{a^3 - 2a^2x + \ \ ax^2} \\
\qquad\qquad\qquad + \ \ a^2x - 2ax^2 + x^3 \\
\qquad\qquad\qquad \underline{\underline{+ \ \ a^2x - 2ax^2 + x^3}}
\end{array}
$$

## Case 3.

The direction given for this case follows from what has before been proved,—that the g. c. m. of two quantities is the product of all the factors common to both.

*Exercise* 1.

The g. c. m. of 8 and 12 is 4; and the g. c. m. of the two compound factors is $a + b$ : $\therefore$ the g. c. m. of the two given quantities is $4(a + b)$.

*Exercise* 2.

$$35x^4 - 35y^4 \doteq 35(x^4 - y^4),$$
$$\text{and, } 63x^2 - 63y^2 \doteq 63(x^2 - y^2).$$

Now the g. c. m. of 35 and 63 is 7, and the g. c. m. of $x^4 - y^4$ and $x^2 - y^2$ is $x^2 - y^2$. $\therefore$ the g. c. m. of the two given quantities is $7(x^2 - y^2)$ or $7x^2 - 7y^2$.

# CHAPTER VIII.

## COMMON MULTIPLES.

### PROBLEM I.

*Demonstration of the Rule.*

By the definition of a *common multiple*, we perceive that any number is a multiple of two others which contains all the factors of each. Thus $abc$ is a common multiple of $ab$ and $ac$. But so also is $a^2bc$, or $abcd$. The least common multiple, then, of the two, will evidently be that which contains the factors common to both only once, along with the other factors of both. But the factors common to both are their g. c. measure. These, therefore, are omitted from the one quantity by dividing it by the g. c. measure, but are retained in the other, and, consequently, appear only once in the product.

Thus, if the quantities are $15abx$ and $20acy$, the g. c. measure, or the factors common to both, being $5a$, if we omit these from the one number, $15abx$, we have $3bx$, which, multiplied into the other number, gives $3bx \times 4 \times 5acy$, or $60abcxy$, a quantity which contains all the factors of each of the given quantities, and those only once.

*Exercise* 5.

G. c. m. $\doteq 2x - y$.

*Exercise* 6.

G. c. m. $\doteq x^2 + 1$.

### PROBLEM II.

*Demonstration of the Rule.*

When we divide one of the two quantities by the g. c. measure of the two, according to Rule 1, we prevent those factors contained in the g. c. measure from appearing twice in the common multiple: and when we again divide the latter and another of the quantities by their g. c. measure

we also prevent any other common factors from appearing twice; and so on; so that the ultimate product is the least common multiple.

The same object is secured by Rule 2 in a different manner.

*Exercises* 1, 2, 3, 4, and 5 should be done by Rule 2.

*Exercise* 6.

The g. c. m. of $x^2-y^2$ and $x^2+2xy+y^2$ is $x+y$.

$$\text{Then } (x^2-y^2)\div(x+y)\doteq x-y.$$

$$(x^2+2xy+y^2)\times(x-y)\doteq x^3+x^2y-xy^2-y^3.$$

The g. c. m. of this and $x^2-2xy+y^2$ is $x-y$.

$$(x^2-2xy+y^2)\div(x-y)\doteq x-y.$$

$$(x^3+x^2y-xy^2-y^3)\times(x-y)\doteq x^4-2x^2y^2+y^4.$$

*Exercise* 7.

The two first quantities are omitted because each of them measures $x^2-4$. We then find that $x^2+4$ and $x^2-4$ have no common measure. Consequently their least common multiple is their product.

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# CHAPTER IX.

## FRACTIONS.

NOTE. There are two ways in which a fraction may be defined. These are sometimes both employed by the same writer, and confounded with each other as if they were identical, whereas they are essentially distinct. Thus 2 thirds may either be defined to be 2 divided by 3, or 2 of the 3 equal parts into which a unit is divided. These two definitions are, no doubt, consistent; but they must be proved to be so and not assumed. In the following demonstrations the latter is taken as the true definition.

### PROBLEM I.

*Demonstration of the Rule.*

If a unit is divided into $n$ equal parts, then 1 unit contains $n$ such parts, and $m$ units contain $mn$ such parts.

That is, $1 \doteq \frac{n}{n}$, and $m \doteq \frac{mn}{n}$. If therefore $m$ and $n$ are given, the numerator of the required fraction will be $mn$.

### Problem II.

*Demonstration of the Rule.*

Since $m \doteq \frac{mn}{n}$, $m+\frac{p}{n} \doteq \frac{mn}{n}+\frac{p}{n}=\frac{mn+p}{n}$, for $mn$ parts added to $p$ parts must make $mn+p$ parts of the same kind, the denominator $n$ merely indicating what kind of parts they are.

In like manner $m-\frac{p}{n} \doteq \frac{mn}{n}-\frac{p}{n}=\frac{mn-p}{n}$.

*Exercise* 6.

$$\begin{array}{r} (3x-4y)\times(4x-3y) \doteq 12x^2-25xy+12y^2 \\ \text{Subtract.........}+\ \ 5xy+12y^2 \\ \hline 12x^2-30xy. \end{array}$$

*Exercise* 7.

$$\begin{array}{r} (8n-3)\times(m-1) \doteq 8mn-3m-8n+3 \\ \text{Subtract...........}8mn-6m \qquad\qquad \\ \hline +3m-8n+3. \end{array}$$

*Exercise* 8.

$$\begin{array}{r} (x-y)\times(x^2+xy+y^2) \doteq x^3-y^3 \\ \text{Subtract.........}x^3+y^3 \\ \hline -2y^3. \end{array}$$

*Exercise* 9.

$$\begin{array}{r} (a^4+ax^3)\times(a^3-x^3) \doteq a^7-ax^6 \\ \text{Add.........}+ax^6-x^7 \\ \hline a^7 \quad * \quad -x^7. \end{array}$$

## PROBLEM III.

*Demonstration of the Rule.*

The fraction $\frac{m}{n}$, as has been said before, means that a unit is divided into $n$ equal parts, and that $m$ such parts constitute the fraction. This definition is alike applicable whether $m$ be less than $n$, greater than $n$, or equal to it. We have now to prove that the fraction $\frac{m}{n} \doteq m \div n$. The quotient $m \div n$ means the $n$th part of $m$: but evidently we shall have the $n$th part of $m$ if we have the $n$th part of all the separate units into which $m$ is divided. But, by our definition of a fraction, the $n$th part of one of those units is $\frac{1}{n}$, and in $m$ units we have $m$ such parts, or $\frac{m}{n}$. $\therefore \frac{m}{n} \doteq m \div n$. Consequently, when $m$ is greater than $n$, if we actually divide $m$ by $n$, we shall have the same quantity that the fraction expresses, only in a different form, viz. in that form required by the problem.

Or, since the object of this problem is the reverse of that of the two preceding problems, that object will be attained by reversing their process: in the two previous problems we multiply by the given denominator; in this we divide by it.

## PROBLEM IV.

*Demonstration of the Rule.*

The truth of this rule depends upon that of the Note which immediately precedes it,—that a fraction is not increased if both terms are multiplied by the same quantity, nor diminished if both are divided by the same quantity, or, more correctly, that a fraction is changed in form but not in value when both terms are multiplied or both divided by the same quantity; and *that* is easily demonstrated thus:—

When the denominator is doubled, it implies that the unit is divided into twice as many parts, and, consequently, that each part is only half the size of one of the original

parts. It therefore requires twice as many such parts to make up the same quantity, and twice as many such parts are obtained by doubling the numerator. In general, if we have the fraction $\frac{a}{b}$ and multiply both terms by $m$, making $\frac{ma}{mb}$, we have $m$ times the previous number of parts, and each part $m$ times smaller, or rather the $m$th part of one of the original parts. Consequently the fraction is equivalent to the original fraction, or $\frac{ma}{mb} \doteq \frac{a}{b}$. Now, if the former of these fractions is obtained from the latter by multiplying both terms by $m$, the latter may be obtained from the former by dividing both terms by $m$. Therefore both the statements of the Note are established.

Such demonstrations, however, should be elucidated to learners by actually taking some object and dividing it.

## Problem V.

*Demonstration of the Rule.*

A fraction is in its lowest terms when its terms have no common measure greater than 1. If, then, the terms of a fraction are both divided by their g. c. m., that is, by all the factors common to both, there remains no other factor common to both, by which they can afterwards be divided; or, in other words, the fraction has been reduced to its lowest terms.

*Exercises.*

| | | |
|---|---|---|
| (1.) G. c. m. $\doteq y$. | (2.) $5m^2n$. | (3.) $137a^2b^3c$. |
| (4.) $a-x$. | (5.) $m+1$. | (6.) $a^2-x^2$. |
| (7.) $a-5$. | (8.) $38(a+b^2)$. | (9.) $5(a-b)$. |

## Problem VI.

The Rule needs no *demonstration* further than that already given under Problem IV, and none of the Exercises *require* detailed *solutions*.

## PROBLEM VII.

*Demonstration of the Rule.*

It scarcely requires proof that $\frac{3}{7}+\frac{2}{7}\doteq\frac{5}{7}$, for the denominator 7 merely indicates what kind of parts they are, while the numerator counts their number. Since, then, 3 pounds and 2 pounds together make 5 pounds, or 3 feet and 2 feet make 5 feet, in like manner 3 sevenths and 2 sevenths will make 5 sevenths. The same explanation will apply to all other fractions having their denominators the same; and, if they have not, they can first be changed into equivalent fractions having the same denominator, by Problem VI, and then their numerators may be added together.

*Exercise* 10.

$$\frac{m+n}{m-n}(m+n \doteq \frac{m^2+2mn+n^2}{m^2-n^2}.$$
$$\frac{m-n}{m+n}(m-n \doteq \frac{m^2-2mn+n^2}{m^2-n^2}.$$

*Exercise* 11.

$$\frac{1}{y^2+2yz+z^2}(y^2-2yz+z^2 \doteq \frac{y^2-2yz+z^2}{y^4-2y^2z^2+z^4}.$$
$$\frac{1}{y^2-z^2}(y^2+z^2 \qquad \doteq \frac{y^2+z^2}{y^4-2y^2z^2+z^4}.$$
$$\frac{1}{y^2-2yz+z^2}(y^2+2yz+z^2 \doteq \frac{y^2+2yz+z^2}{y^4-2y^2z^2+z^4}.$$

NOTE 1. The reason that we regard the sign as attached to the numerator, is, that it really belongs to it and not to the denominator. When we say "$-5$ pounds" the sign $-$ belongs to the 5 and not to the word "pounds" expressing the denomination: so, when we say "$-5$ sevenths" or "$-\frac{5}{7}$," the sign $-$ belongs to the 5 and not to the 7, which merely expresses what kind of parts they are. When the numerator is a compound quantity preceded by a negative sign, the negative sign then applies to the whole numerator, exactly as if it were enclosed in a vinculum. Upon removing the vinculum, therefore, and taking the terms

separately, we must change all the signs by Rule 4 of Chapter IV.

*Exercise* 14.

$$\frac{a^2+2ax+x^2}{a-x} (a+x \doteq \frac{+a^3+3a^2x+3ax^2+x^3}{a^2-x^2}.$$

$$-\frac{a^2-2ax+x^2}{a+x} (a-x \doteq \frac{-a^3+3a^2x-3ax^2+x^3}{a^2-x^2}.$$

*Exercise* 15.

$$\frac{2a}{a^2+2ax+x^2} (a-x \doteq \frac{2a^2-2ax}{a^3+a^2x-ax^2-x^3}.$$

$$\frac{2x}{a^2-x^2} (a+x \doteq \frac{2ax+2x^2}{a^3+a^2x-ax^2-x^3}.$$

### PROBLEM VIII.

The *principle* is the same as in Addition.

*Exercise* 11.

$$\frac{2a}{a^2+2ax+x^2} (a-x \doteq \frac{2a^2-2ax}{a^3+a^2x-ax^2-x^3}.$$

$$\frac{2x}{a^2-x^2} (a+x \doteq \frac{2ax+2x^2}{a^3+a^2x-ax^2-x^3}.$$

### PROBLEM IX.

*Demonstration of the Rule.*

When we say £7, and when we say £$\frac{2}{3}$, or 7 pounds and $\frac{2}{3}$ of a pound, we, in both instances, express the number of pounds; but, when that number is fractional, we vary the mode of expression. Thus 7 pounds and $\frac{21}{3}$ of a pound are identical, the 7 and the $\frac{21}{3}$ alike expressing the number of pounds. So 7 fives, and $\frac{2}{3}$ of 5, alike express a certain number of fives. But 7 fives is the same as 7 times 5, or $5 \times 7$ : so $\frac{2}{3}$ of 5 is the same as $5 \times \frac{2}{3}$. One idea is attached to both expressions, although the form of expres-

sion is different.* In like manner $\frac{2}{3}$ of $\frac{5}{7}$ is the same as $\frac{5}{7} \times \frac{2}{3}$.

When, therefore, we are called upon to multiply a fraction by a fraction, the clearest idea of what is intended is obtained by changing the form of expression. When it is said "multiply $\frac{5}{7}$ by $\frac{2}{3}$" it is meant "take $\frac{2}{3}$ of $\frac{5}{7}$." Having then ascertained the true meaning of "multiplying a fraction by a fraction," we may next investigate how the object intended is to be attained. It is evident that we shall readily obtain *two* thirds of $\frac{5}{7}$ if we can obtain *one* third of it; and it is equally evident that we may easily obtain one third of *five* sevenths if we can find one third of *one* seventh.

In order to have 1 seventh we divide the unit into seven equal parts, and take one of them; and 1 third of that one

will be obtained by dividing *it* into three equal parts. But if we divide each of the sevenths into three parts, we have 7 threes, or 21 parts altogether, and one of these is the 21st part of the unit. Therefore $\frac{1}{3}$ of $\frac{1}{7} = \frac{1}{21}$. But $\frac{1}{3}$ of $\frac{5}{7}$ is obtained by taking $\frac{1}{3}$ of $\frac{1}{7}$ five times, that is five times $\frac{1}{21}$, or $\frac{5}{21}$. Again, if *one* third of $\frac{5}{7} = \frac{5}{21}$, *two* thirds must be twice as much. But twice 5 makes 10: therefore twice $\frac{5}{21} = \frac{10}{21}$. Consequently $\frac{2}{3}$ of $\frac{5}{7} = \frac{10}{21}$.

Now the 10 was obtained by multiply 5 by 2; and the 21, by multiplying 3 by 7. Hence the rule.

The process of *cancelling* the common factors, as directed in Note 1, is merely anticipating the reduction of the fraction to its lowest terms. It avoids the unnecessary labour of first multiplying and then dividing.

## PROBLEM X.

*Demonstration of the Rule.*

Since $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ by the preceding Problem, and since Division is just the reverse of multiplication, it follows that $\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$. But this is just what we should have obtained

* Wood's Algebra (30).

by multiplying $\frac{ac}{bd}$ by $\frac{d}{c}$. Now let $ac$ be expressed by the single letter $m$, and $bd$ by $n$, it follows that $\frac{m}{n} \div \frac{c}{d} \doteq \frac{m}{n} \times \frac{d}{c}$.

*Exercise* 7.

$$\frac{a^2+ab}{a-b} \times \frac{a^2-2ab+b^2}{a^3+ab^2} \doteq \frac{a+b}{1} \times \frac{a-b}{a^2+b^2},$$

the common factors $a$ and $(a-b)$ being cancelled.

*Exercise* 8.

$$\frac{x+y}{4x+6y} \times \frac{2x+3y}{2x-3y} \doteq \frac{x+y}{2} \times \frac{1}{2x-3y},$$

the common factor $(2x+3y)$ being cancelled.

*Exercise* 9.

$$\left(x-\frac{y}{a}\right) \div \frac{x}{a} \doteq \frac{ax-y}{a} \times \frac{a}{x} = \frac{ax-y}{1} \times \frac{1}{x}, \text{ or}$$

$$\left(x-\frac{y}{a}\right) \div \frac{x}{a} = \left(x \div \frac{x}{a}\right) - \left(\frac{y}{a} \div \frac{x}{a}\right) = \left(\frac{x}{1} \times \frac{a}{x}\right) - \left(\frac{y}{a} \times \frac{a}{x}\right).$$

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# CHAPTER X.

## INVOLUTION.

### *Demonstration of the Rule.*

Since $(ab)^3 \doteq ab.ab.ab = aaabbb = a^3b^3$, and since the same reasoning may be applied to any power or to any number of factors, it follows that—to raise a number composed of several factors to any power, we raise each factor separately to that power.

Now, if one of the factors is a particular number, as 5, the same principle holds good: that is—we raise the given number separately to the required power, and the literal factors separately to the same power. Hence the rule for the *co-efficient.*

*Next let* any of the factors of the given quantity be itself

a power, as $a^2$, and let it be required to raise that to some given power, say the third. Since $a^2 \doteq aa$, $(a^2)^3 \doteq (aa)^3 = aa \times aa \times aa = aaaaaa = a^6$. But 6, the exponent of the power found, is the same as the number of times that $a$ is expressed in *aaaaaa*, which, again, was derived from the three factors, each containing $a$ twice, making 3 times 2, or, in other words, the exponent of the given literal factor multiplied by the exponent of the proposed power.

The rule for *the sign* is evident; for, if the given quantity have the sign $+$, it will continue to have $+$ through any number of multiplications into itself. But if the given quantity have the sign $-$, it becomes $+$ for the second power (since $-$ into $-$ gives $+$), then $-$ for the third power (since $+$ into $-$ gives $-$), and so on.

## Problem II.

The rule requires no *demonstration*.

### *Exercise* 4.

2d power, $25x^2 - 10x + 1$.
3d ........, $125x^3 - 75x^2 + 15x - 1$.

### *Exercise* 5.

2d power, $4m^2 - 12mn + 9n^2$.
3d ........, $8m^3 - 36m^2n + 54mn^2 - 27n^3$.
4th ........, $16m^4 - 96m^3n + 216m^2n^2 - 216mn^3 + 81n^4$.

### *Exercise* 6.

2d power, $x^2 - 2xy + y^2$.
3d ........, $x^3 - 3x^2y + 3xy^2 - y^3$.
4th ........, $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.
5th ........, $x^5 - 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 - y^5$.
6th ........, $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.

### *Exercise* 8.

2d power, $16a^4 + 40a^3 + 25a^2$.

### *Exercise* 10.

2d power, $x^2 - \frac{4}{3}x + \frac{4}{9}$.

PROBLEM III.

The rule is the immediate result of that for the multiplication of fractions.

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## CHAPTER XI.

## EVOLUTION.

### PROBLEM I.

Evolution being the reverse of Involution, the rule for the former follows directly from that for the latter, in the case of simple quantities.

### PROBLEM II.

*Demonstration of the Rule.*

If the quantity, whose root is to be extracted, is a complete square, we shall have its root by observing the way in which the square is made up. If it consist of only three terms it must be the square of a binomial; for the square of every quantity containing more than two terms, contains, itself, more than three. Let us, then, first of all, observe what parts the square of a binomial consists of.

$$(a \pm b)^2 \doteq a^2 \pm 2ab + b^2.$$

That is, the first and last terms of the square are the squares of the two terms of the root, and the second term of the square is twice the product of the two terms of the root (including their respective signs). This will be the case whether the terms ($a$ and $b$) are single letters or numbers or made up each of several factors. Thus $(4x \pm 3y)^2 \doteq (4x)^2 \pm 2.4x.3y + (3y)^2$, since, in this case, we have $4x$ substituted for $a$ and $3y$ for $b$.

If, then, we have a regular square of three terms, let us, in order to find the root, arrange the terms in their proper order, which is evidently according to the powers of the *letters*. Then the square root of the first term, from what

has been said, will be the first term of the root. Now the second term of the square being twice the product of the first and second terms of the root, it follows that, if we divide the second term of the square by twice the first term of the root just found, we have the second term of the root. Thus, let it be required to find the square root of $16x^2 - 24xy + 9y^2$. We begin with the first term $16x^2$, the square root of which is $4x$, by Pr. I, and *that* we take as the first term of the root. Rejecting, then, this term from the power, by subtracting it, and retaining (or bringing down) the other terms, we know that the next term, $-24xy$,

$$\begin{array}{r|l} & 16x^2 - 24xy + 9y^2 \,(4x - 3y \\ & 16x^2 \\ \hline 8x - 3y) & -24xy + 9y^2 \\ & -24xy + 9y^2 \\ \hline \end{array}$$

must be made up of twice the product of $4x$ and the second term, or of the product of $8x$ and that term. Consequently the second term $\doteq (-24xy) \div (+8x) = -3y$. We prove this by multiplying $8x$ by $-3y$, and also multiply $-3y$ by $-3y$, since the third term should be the square of $-3y$. If these two products, then, agree with the two remaining terms of the square, leaving no remainder when subtracted, the work is finished and the given quantity is proved to be a complete square.

Next, if the given square consists of more than three terms, the root, consequently, consisting of more than two, we continue the operation in the same way, taking the first two terms of the root as one quantity, and using that to find the third, precisely as we used the first to find the second. Thus, in the Example in the Algebra, having found the two first terms, $a^2 - 2ab$, in the manner previously described, we take the whole compound quantity $(a^2 - 2ab)$, and regarding it now as one term, we obtain another term, $+b^2$, in the same manner in which we obtained the second term. As a first step to this the square of $(a^2 - 2ab)$ has already been subtracted from the given power in two parts, first $a^4$, and then $-4a^3b + 4a^2b^2$, after which we have merely to divide the remainder by $2(a^2 - 2ab)$, or $(2a^2 - 4ab)$, to find the third term, $+b^2$.

If the given power consists of more than five terms, we may proceed in the same manner to find the other term or terms of the root.

If the given quantity is not an exact square, we may, by the same method, take the nearest root we can obtain, annexing + or − to it according as a positive or a nega-

tive quantity remains when the square of the root found has been subtracted. Or, if desirable, the process may be continued till the root acquire the form of a regular series: but that is seldom, if ever, of any use unless the first term of the root is greater than the second and subsequent terms, so that the series may *converge*; that is—that the terms may become continually less and less till, by going on, we have it in our power to arrive at a term less than any assignable quantity, or such that all the subsequent terms put together shall be less than any assignable quantity.

*Exercise* 3.

$$x^4 - 2x^2y^2 + y^4 + 2x^2z^2 - 2y^2z^2 + z^4(x^2 - y^2 + z^2$$
$$x^4$$
$$2x^2 - y^2) - 2x^2y^2 + y^4$$
$$- 2x^2y^2 + y^4$$
$$2x^2 - 2y^2 + z^2) \qquad + 2x^2z^2 - 2y^2z^2 + z^4$$
$$+ 2x^2z^2 - 2y^2z^2 + z^4.$$

*Exercise* 4.

$$a^6 - 4a^5 - 2a^4 + 12a^3 + 9a^2(a^3 - 2a^2 - 3a.$$
$$a^6$$
$$2a^3 - 2a^2) - 4a^5 - 2a^4$$
$$- 4a^5 + 4a^4$$
$$2a^3 - 4a^2 - 3a) - 6a^4 + 12a^3 + 9a^2$$
$$- 6a^4 + 12a^3 + 9a^2.$$

*Exercise* 5.

$$4x^2 - 12xy + 9y^2 + 16xz - 24yz + 16z^2(2x - 3y + 4z.$$
$$4x^2$$
$$4x - 3y) - 12xy + 9y^2$$
$$- 12xy + 9y^2$$
$$4x - 6y + 4z) \qquad + 16xz - 24yz + 16z^2$$
$$+ 16xz - 24yz + 16z^2.$$

*Exercise* 6.

$$
\begin{array}{l}
\quad 25a^2+10ab-2ac+b^2-\frac{2}{5}bc+\frac{1}{25}c^2(5a+b-\frac{1}{5}c \\
\quad 25a^2 \\
10a+b)+10ab-2ac+b^2 \\
\qquad\quad +10ab \quad * \quad +b^2 \\
10a+2b-\frac{1}{5}c) \quad -2ac \quad * \quad -\frac{2}{5}bc+\frac{1}{25}c^2 \\
\qquad\qquad\qquad\quad -2ac \quad * \quad -\frac{2}{5}bc+\frac{1}{25}c^2.
\end{array}
$$

*Exercise* 7.

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad (3x^3-2x^2+x-4 \\
\quad 9x^6-12x^5+10x^4-28x^3+17x^2-8x+16 \\
\quad 9x^6 \\
6x^3-2x^2)-12x^5+10x^4 \\
\qquad\qquad -12x^5+\ 4x^4 \\
6x^3-4x^2+x) \quad +\ 6x^4-28x^3+17x^2 \\
\qquad\qquad\qquad +\ 6x^4-\ 4x^3+\ 1x^2 \\
6x^3-4x^2+2x-4) \qquad -24x^3+16x^2-8x+16 \\
\qquad\qquad\qquad\qquad\quad -24x^3+16x^2-8x+16.
\end{array}
$$

---

# CHAPTER XII.

## PROPORTION.

**NOTE.** In the demonstrations of this Chapter we are obliged to anticipate the subject of Equations; and, in fact, the latter subject ought, to a certain extent, to be taught first, although it is placed after Proportion in the Course. The reason that it is so placed, is, that a knowledge of Proportion is necessary for many of the exercises in Equations and of the questions which follow. The pupils should go over Equations first, as far as Exercise 35, page 49, then return to Proportion, and after finishing that subject resume Equations.

*Demonstration of Theorem I.*

If $a : b :: c : d$, then, by Def. 1, $\frac{a}{b} = \frac{c}{d}$. Multiplying both sides of the equation by $bd$, we have $ad = bc$.

*Demonstration of Theorem II.*

If $a : b :: b : c$, then, $\frac{a}{b} = \frac{b}{c}$, and $\therefore ac = b^2$.

*Demonstration of Theorem III.*

If $ad = bc$, then, dividing both sides of the equation by $bd$, we have $\frac{a}{b} = \frac{c}{d}$, or $\frac{c}{d} = \frac{a}{b}$, and, consequently, by Def. 1, $a : b :: c : d$, or $c : d :: a : b$.

If we divide both sides of the first equation by $ac$, we have $\frac{d}{c} = \frac{b}{a}$, or $\frac{b}{a} = \frac{d}{c}$, and $d : c :: b : a$, or $b : a :: d : c$.

*Demonstration of Theorem IV.*

If $a : b :: c : d$, then, by Th. I, $bc = ad$. Dividing both sides by $ac$, we have $\frac{b}{a} = \frac{d}{c}$. Consequently $b : a :: d : c$.

*Demonstration of Theorem V.*

If $a : b :: c : d$, then, by Th. I, $ad = bc$. Dividing both sides by $cd$, we have $\frac{a}{c} = \frac{b}{d}$. $\therefore a : c :: b : d$.

*Demonstration of Theorem VI.*

If $a : b :: c : d$, then $ad = bc$. Multiplying both sides of the equations by $m$, we have $adm = bcm$. That is, $a \times md = b \times mc$, or $ma \times d = mb \times c$, giving, by Theorem III, the two propositions, $a : b :: mc : md$, and $ma : mb :: c : d$.

We may also express our equation thus, $ma \times d = mc \times b$, or $a \times md = c \times mb$, giving two proportions thus, $ma : b :: mc : d$, and $a : mb :: c : md$.

Again, returning to our original equation, $ad = bc$, and multiplying both sides by $mn$, we have $admn = bcmn$; that

is, $ma \times nd = mb \times nc$, or $ma \times nd = nb \times mc$. $\therefore ma : mb :: nc : nd$, or $ma : nb :: mc : nd$.

If we had used $\frac{1}{m}$ and $\frac{1}{n}$ throughout the preceding steps, instead of $m$ and $n$, we should have found, in like manner, that $a : b :: \frac{c}{m} : \frac{d}{m}$, that $\frac{a}{m} : \frac{b}{m} :: c : d$, that $\frac{a}{m} : b :: \frac{c}{m} : d$, that $a : \frac{b}{m} :: c : \frac{d}{m}$, that $\frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$, and that $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.

*Demonstration of Theorem VII.*

If $a : b :: c : d$, then, by Th. I, $ad \doteq bc$. Raising both members of the equation to any power, say the $n$th power, we have $a^n d^n = b^n c^n$. Hence, by Th. III, $a^n : b^n :: c^n : d^n$.

If, instead of raising both members of the equation to the $n$th power, we had extracted the $n$th root, we should have had $\sqrt[n]{a}\sqrt[n]{d} = \sqrt[n]{b}\sqrt[n]{c}$, and $\sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$.

*Exercise 2.*

Ans. $3a : 5b :: 2c^2 : 6d^2$.

PROBLEM I.

*Demonstration of the Rule.*

If $a : b :: c : d$, then $ad \doteq bc$. Consequently, according as we divide both sides by $d$, $c$, $b$, or $a$, we have $a = \frac{bc}{d}$, $b = \frac{ad}{c}$, $c = \frac{ad}{b}$, or $d = \frac{bc}{a}$.

*Exercise 3.*

Putting $p$ for the piece of ground, then, in one day,

The gardener digs $\frac{1}{10} p = \frac{12}{120} p$
The apprentice digs $\frac{1}{12} p = \frac{10}{120} p$, and
Both together dig $\frac{22}{120} p = \frac{11}{60} p$.
Then $\frac{11}{60} p : p :: 1$ day $: \frac{60}{11} = 5\frac{5}{11}$ days.

*Exercise* 4.

Putting $q$ for the whole quantity of water which the cistern holds, we find, from the data, that in one hour,

$$\begin{array}{ll} \text{1st pipe discharges} & \frac{1}{24}q = \frac{15}{360}q. \\ \text{2d} \dots\dots\dots\dots\dots\dots & \frac{1}{20}q = \frac{18}{360}q. \\ \text{3d} \dots\dots\dots\dots\dots\dots & \frac{1}{18}q = \frac{20}{360}q. \\ \hline \text{All together} \dots\dots\dots\dots\dots & \frac{53}{360}q. \end{array}$$

Then $\frac{53}{360}q : q :: 1 \text{ hour} : \frac{360}{53} = 6\frac{42}{53}$ hours.

*Exercise* 5.

Put $c$ for the cheese: then, in one day,

$$\begin{array}{ll} \text{Both together eat} & \frac{1}{14}c = \frac{18}{252}c, \\ \text{The wife alone eats} & \frac{1}{36}c = \frac{7}{252}c. \\ \hline \therefore \text{ the husband eats} & \frac{11}{252}c. \end{array}$$

Hence $\frac{11}{252}c : c :: 1 \text{ day} : \frac{252}{11} = 22\frac{10}{11}$ days.

PROBLEM II.

*Demonstration of the Rule.*

If $a : b :: b : c$, then, by Th. II, $ac = b^2$. $\therefore b = \sqrt{ac}$.

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## CHAPTER XIII.

### EQUATIONS.

#### DEFINITIONS.

NOTE TO DEF. 3. No single simple equation containing only one unknown quantity can be impossible. In that case the unknown quantity can always have a value given to it that will satisfy the conditions of the equation, that value being either positive, negative, or $= 0$.

Any number of simple equations too are always possible (that is, not inconsistent with each other) so long as there are not more equations than unknown quantities, or rather, *so long as no* number of the given equations can be taken

in which there are fewer unknown quantities than equations. Thus, the following equations are inconsistent although there are not altogether more equations than unknown quantities: for, if we select the three first equations, *they* contain only two unknown quantities.

$$\begin{aligned} 3x+2y&=12 \\ 4x+3y&=17 \\ 5x+4y&=23 \\ w+x+y+z&=16. \end{aligned}$$

*Demonstration of the Rules.*

The first four rules are self-evident. The mark $=$, indicating equality, appears to have been originally the emblem of a balance, being still found, in that capacity, among the marks used for the signs of the zodiac; and nothing affords a better illustration, than a balance, of these four rules.

Rule 5 is derived from rule 3; for, when two equals are raised both to the same power, they are, at each successive step, multiplied by equals: consequently the successive products are equal, including the final product.

Rule 6 follows from rule 5. There is a qualification, however, to this rule, since two quantities, in themselves equal but with different signs, have their even powers the same, in both cases with positive signs; and, consequently, every positive quantity has two roots when the index of the root is even.* In such cases, then, it is only the positive root that is equal to the positive, and the negative to the negative.

Rule 7 is self-evident.

Rule 8 is derived from Rules 1 and 2; for, if $a-b \doteq c$, add $b$ to each side, and $a \doteq c+b$; or, if $a+b \doteq c$, take $b$ from each side, and $a \doteq c-b$.

Rule 9 is self-evident; for, if $a \doteq b$, $b \doteq a$, whether $a$ and $b$ be positive or negative, and whether simple or compound quantities.

Rule 10 may also be regarded as self-evident; for, if $+a \doteq +b$, $-a \doteq -b$. But it may be deduced from Rules 8 and 9, if we choose: for we may transpose each term, changing the sign, by Rule 8, and then re-transpose all

* In the Elementary Course no notice is taken of impossible quantities, which should be reserved for a more advanced stage. If these were taken into account, we should find that every quantity has as many roots as the number which is the index of the *root.*

the terms without changing the signs, by Rule 9. Thus, if $a-b\doteq x-y$, then $-x+y\doteq -a+b$ by Rule 8, and $-a+b\doteq -x+y$, by Rule 9.

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# CHAPTER XIV.

## SIMPLE EQUATIONS.

### Problem I.

*Exercise* 18.

Multiplying by the l. c. mult. of 6 and 10, viz. 30,

$$25x-21x-1920\doteq 0.$$
$$4x\doteq 1920.$$

*Exercise* 19.

Multiply by l. c. m. $=12$.

$$2x-3x-108\doteq 4x-6x.$$

*Exercise* 20.

Multiply by l. c. m. $=336$.

$$320x-25200\doteq 140x+105x.$$
$$75x\doteq 25200.$$

*Exercise* 21.

Multiply by l. c. m. $=15$.

$$3y+6\doteq 10y-30-90.$$
$$126\doteq 7y.$$

*Exercise* 22.

Multiply by l. c. m. $\doteq 12$.

$$9x-99-144\doteq 126-6x-16x-90.$$
$$31x\doteq 279.$$

*Exercise* 23.

Multiply by $6y$.

$$180\doteq 240-5y.$$
$$5y\doteq 60.$$

*Exercise* 24.

Multiply by $6z$.

$$48z-210 \doteq 84+42.$$
$$48z \doteq 336.$$

*Exercise* 25.

Multiply by $4-y$.

$$y-28+7y \doteq 96-24y.$$
$$32y \doteq 124.$$

*Exercise* 26.

Multiply by 4.

$$10x-3x-5 \doteq 44.$$
$$7x \doteq 49.$$

*Exercise* 27.

Multiply by 21.

$$105x+49x+14-15x+18=588.$$
$$139x \doteq 556.$$

*Exercise* 28.

Multiply by 36.

$$27x-54-36x+96 \doteq 384-216-16x.$$
$$27x-36x+16x \doteq 54-96+384-216.$$
$$7x \doteq 126.$$

*Exercise* 29.

Squaring both sides,

$$7x \doteq 1225.$$

*Exercise* 30.

$$5\sqrt{x} \doteq 15.$$
$$\sqrt{x} \doteq 3,$$

*Exercise* 31.

Squaring both sides,

$$y-35 \doteq 225.$$

*Exercise* 32.

Cubing both sides,

$$4z+3 \doteq 27.$$

*Exercise* 33.

Squaring both sides,

$$19x+13 \doteq 59-4x.$$

*Exercise* 34.

Squaring both sides,

$$y+9 \doteq 1+2\sqrt{y}+y.$$
$$4 \doteq \sqrt{y}.$$

*Exercise* 35.

Reduce the fraction to its lowest terms, or divide its numerator by its denominator, and we have

$$2(\sqrt{5z}-3)-2=\sqrt{5z}-3.$$
$$\therefore \sqrt{5z}-3 \doteq 2,$$
$$\text{and } \sqrt{5z} \doteq 5.$$

*Exercise* 36.

Making the product of the extremes equal to that of the means, we have

$$20x=9x+180.$$

*Exercise* 37.

$$10+x \doteq 8x-18.$$

*Exercise* 38.

$$17-4x \doteq \frac{75+10x}{3}-10x.$$
$$17+6x \doteq \frac{75+10x}{3}.$$
$$51+18x \doteq 75+10x.$$
$$8x \doteq 24.$$

## Problem II.

The rule requires no demonstration, being merely an application of principles previously established.

In the following solutions of Exercises the operation will, in general, be given only by one of the three methods, the easiest being selected; but it will be proper that the Teacher should see the pupils perform a few of them by all the methods.

Note. When a value is found for one of the two unknown quantities, from the two given equations combined, we have ascertained that, whatever may be the value of the other unknown quantity, the value, which we have found for the first, must satisfy both equations. When we come to substitute that value for the first in one of the two given equations, we obtain a value for the other unknown quantity, which, we know, *must* satisfy that one of the given equations from which it has been obtained. But it does not *evidently* follow that it will satisfy the other. It will do so, however, in all cases in which the given equations are possible, or the conditions not inconsistent. For it can easily be proved, in the case of simple equations, that *no other* value than that found for the first unknown quantity can satisfy both equations, and that *only* the value found for the other unknown quantity can satisfy the equation employed to determine it. Consequently, if that value do not satisfy the other given equation, there is no value that will do so. In equations of higher orders, when more values than one are found for the unknown quantities, from one equation, *all* these values may not hold good when tried with the other equation. See the Note to Ex. 5 of Pr. III of Chap. XVI, in this volume.

*Exercise* 1.

First adding the two given equations together, then subtracting the second from the first, we have

$$2x=32, \text{ and}$$
$$2y=18.$$

Note. The method of resolving this exercise should be particularly observed. It will often be required afterwards.

*Exercise* 2.

Subtract Eq. 2 from Eq. 1.

$$7y \doteq 42, \text{ and } y \doteq 6.$$

Inserting that value for $y$ in Eq. 2, we obtain

$$6x - 18 = 30, \text{ and } \therefore 6x = 48.$$

*Exercise* 3.

Multiply Eq. 1 by 4, and subtract Eq. 2 from the product.

*Exercise* 4.

Multiply Eq. 1 by 6, and Eq. 2 by 7. Then subtract.

*Exercise* 5.

Multiply Eq. 1 by 3, and Eq. 2 by 2.

*Exercise* 6.

Subtract Eq. 1 from Eq. 2.

*Exercise* 7.

Multiply Eq. 2 by 3, and from the product subtract Eq. 1.

*Exercise* 8.

Multiply Eq. 1 by 3, and Eq. 2 by 2, and add.

*Exercise* 9.

Multiply Eq. 2 by $y$. Then

$$x \doteq \tfrac{2}{3}y.$$

Insert this value, for $x$, in Eq. 1.

$$\tfrac{2}{3}y + y \doteq 20.$$

*Exercise* 10.

Divide Eq. 1 by Eq. 2, and we have

$$x + y = 21.$$

Proceed with Eq. 2, and that just found, as in Exercise 1.

*Exercise* 11.

Clear both equations of fractions by multiplying each by 6.

$$45y + 80z \doteq 3{,}960, \ldots\ldots\ldots\ldots (3).$$
$$50y + 33z \doteq 2{,}388, \ldots\ldots\ldots\ldots (4).$$

Subtract Eq. 4 from Eq. 3.

$$-5y + 47z \doteq 1{,}572, \ldots\ldots\ldots\ldots (5).$$

Multiply Eq. 5 by 10.

$$-50y + 470z \doteq 15{,}720.$$

To this add Eq. 4.

$$503z \doteq 18{,}108, \text{ and } z \doteq 36.$$

Insert this value, for $z$, in Eq. 5.

*Exercise* 12.

Multiply Eq. 1 by 6 and Eq. 2 by 2, and simplify the results.

$$+4x + 3y \doteq 48, \ldots\ldots\ldots\ldots (3).$$
$$-3x + 5y \doteq 22, \ldots\ldots \ldots\ldots (4).$$

Multiply Eq. 3 by 3 and Eq. 4 by 4.

$$+12x + \ 9y \doteq 144.$$
$$-12x + 20y \doteq \ 88.$$

Add these two equations together.

$$29y \doteq 232, \text{ and } y \doteq 8.$$

Insert this value of $y$ in Eq. 3.

$$4x + 24 \doteq 48.$$

*Exercise* 13.

Multiply Eq. 1 by 20, and Eq. 2 by 6, and simplify the results.

$$6x + 55z \doteq 128, \ldots\ldots\ldots\ldots (3).$$
$$34x + 15z \doteq 132, \ldots\ldots\ldots\ldots (4).$$

Multiplying Eq. 3 by 3, and Eq. 4 by 11, and subtracting, we find

$$356x = 1068.$$

*Exercise* 14.

From the proportion given we obtain

$$20y - 21x = 0.$$

Multiplying Eq. 1 by 5,

$$20y - 15x \doteq 60$$
$$\therefore 6x \doteq 60.$$

*Exercise* 15.

From the two given proportions we have

$$x + y = 3x - 3y, \text{ and}$$
$$9x + 12 = 30y - 20.$$

Simplifying and arranging,

$$2x \doteq 4y, \text{ or } x = 2y,$$
$$\text{and } 9x - 30y = -32.$$
$$\text{That is, } 18y - 30y \doteq -32;$$
$$\text{or } 12y \doteq 32.$$

*Exercise* 16.

Multiply Eq. 1 by 6, and there results

$$15x + 39 - 8y + 3x + 3 = 54 + 14x - 6y,$$
$$\text{or } 4x - 2y = 12, \ldots\ldots\ldots\ldots (3).$$

From the given proportion we have

$$\frac{7x+49}{2} = 8x + \frac{3y-8}{2}, \text{ or}$$
$$7x + 49 = 16x + 3y - 8.$$
$$\text{That is, } 9x + 3y \doteq 57, \ldots\ldots\ldots\ldots (4).$$

Multiply Eq. 3 by 3, and Eq. 4 by 2, and add.

$$30x \doteq 150.$$

## Problem III.

*No demonstration* is required.

*Exercise* 1.

Subtract Eq. 2 from Eq. 1.

$$2z \doteq 2.$$

Subtract Eq. 3 from Eq. 1.

$$2y \doteq 18.$$

Add Eq. 2 to Eq. 3.

$$2x \doteq 24.$$

*Exercise* 2.

Multiply Eq. 2 by 3.

$$9x - 6y + 3z \doteq 60.$$

Add this to Eq. 1.

$$11x - 4y \doteq 78, \ldots\ldots\ldots\ldots (4).$$

Add Eq. 2 to Eq. 3.

$$7x - 6y \doteq 22, \ldots\ldots\ldots\ldots (5).$$

Multiply Eq. 4 by 3, and Eq. 5 by 2, and subtract.

$$19x \doteq 190, \text{ or } x \doteq 10.$$

Insert 10 for $x$ in Eq. 4, and so on.

*Exercise* 3.

Multiply Eq. 3 by 2.

$$x + \tfrac{2}{3}y - \tfrac{1}{2}z \doteq -2, \ldots\ldots\ldots\ldots (4).$$

Subtract Eq. 4 from Eq. 1, and Eq. 1 from Eq. 2.

$$-1\tfrac{2}{3}y + 1\tfrac{1}{2}z \doteq 10, \ldots\ldots\ldots\ldots (5).$$
$$+3y + 2z \doteq 76, \ldots\ldots\ldots\ldots (6).$$

Multiply Eq. 5 by 4, and Eq. 6 by 3.

$$-6\tfrac{2}{3}y + 6z \doteq 40.$$
$$+9y + 6z \doteq 228.$$
$$\text{Hence } 15\tfrac{2}{3}y \doteq 188, \text{ or } y = 12.$$

Insert 12 for $y$ in Eq. 6.

$$2z \doteq 40.$$

*Exercise* 4.

Subtract Eq. 1 from Eq. 3.

$$y - z = 2.$$

Proceed with this and Eq. 2 as in Exercise 1 of Pr. II.

*Exercise* 5.

Multiply the first Eq. by 3 and the third by $z$.

$$x - y + 3z \doteq 66, \ldots\ldots\ldots\ldots(4).$$
$$x - y - 3z \doteq 0, \ldots\ldots\ldots\ldots(5).$$

Subtract, and we have

$$6z = 66, \text{ or } z = 11.$$

Double Eq. 2.

$$2x + y + z = 32.$$

In this and Eq. 5, insert 11 for $z$.

$$2x + y \doteq 21.$$
$$x - y \doteq 33.$$

Add these two equations.

$$3x \doteq 54.$$

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## CHAPTER XV.

### QUESTIONS PRODUCING SIMPLE EQUATIONS.

*Exercise* 2.

$$\frac{x}{2} - \frac{x}{3} \doteq 17.$$

*Exercise* 3.

$$x + 15 \doteq 2x - 10.$$

*Exercise* 4.

Put $x$ for the number who voted for Jones: then $x + 75$ will be the number who voted for Smith.

$$x \doteq \tfrac{4}{5}(x + 75).$$

### *Exercise* 6.

Let $x$ represent what he gave to the woman; $2x$, what he gave to her husband; $\frac{2x}{3}$, to the oldest child.

$$2x+x+\frac{2x}{3}+1 \doteq 23.$$

### *Exercise* 7.

Let the time taken in walking from $A$ to $C$ be $x$ hours. That from $A$ to $B$ will be $x+\frac{1}{3}$; that from $B$ to $C$, $x-\frac{5}{6}$.

$$x+(x+\tfrac{1}{3})+(x-\tfrac{5}{6}) \doteq 6.$$

Hence we find $x=2\frac{1}{6}$, $x+\frac{1}{3}=2\frac{1}{2}$, $x-\frac{5}{6}=1\frac{1}{3}$. Multiplying these times by 3, we have the distances, $6\frac{1}{2}$, $7\frac{1}{2}$, and 4.

### *Exercise* 8.

Put $x=$ rate per hour.

$$\text{Then } 12x \doteq 10 \times (x+3).$$

### *Exercise* 9.

Let $x$ be the sum each began with.

$$x+2\tfrac{1}{2} \doteq 3(x-1\tfrac{1}{2}).$$

### *Exercise* 10.

$$9+x+\frac{x}{9} \doteq 59.$$

### *Exercise* 11.

Let $500+x$, and $500-x$ be the two parts.

$$9 \times (500+x) \doteq 13(500-x)+970.$$
$$\text{or, } 9 \times 500+9x \doteq 13 \times 500-13x+970.$$
$$\therefore 22x \doteq 4 \times 500+970.$$

### *Exercise* 12.

Let $x$ be the one part: then $100-x$ will be the other.

$$\frac{x}{2}+\frac{100-x}{7} \doteq 20.$$

*Exercise* 13.

Our three given equations are, $x+y=11$, $x+z=12$, and $y+z=13$.

Add these three equations together and halve the sum.

$$x+y+z \doteq 18.$$

From this subtract the three equations separately.

*Exercise* 14.

$$x \times 24 \doteq (x+7) \times 20.$$

*Exercise* 15.

Let the second ride $x$ hours before he overtakes the first, which he will do at the distance of $7x$ miles from the point of departure. But by that time the first has walked $(x+2)$ hours, and consequently $3(x+2)$, or $(3x+6)$ miles.

$$\therefore 3x+6 \doteq 7x.$$

*Exercise* 16.

Let $x$ be the time of delay. Then, if the rider overtake the walker at the 14th milestone, he must have ridden 2 hours, and the other must have walked $(x+2)$ hours.

$$\therefore (x+2) \times 3 \doteq 14.$$

*Exercise* 17.

If $4x$ represent the smaller number, $5x$ will represent the greater, for $4x : 5x :: 4 : 5$. We have, therefore,

$$5x+11 = 2 \times (4x-5).$$

*Exercise* 18.

Let $x$ be the man's age at the time supposed. His wife's age *then* will be $x-20$. Therefore,

$$x-20 \doteq \tfrac{3}{4}x.$$

*Exercise* 19.

$$x+6 : y+6 :: 4 : 5.$$
$$x-4 : y-4 :: 2 : 3.$$

*Exercise* 20.

Let $x$ denote the number of women; then the number of children will be expressed by $200-x$.

$$\therefore x \times 20 + (200-x) \times 2\tfrac{1}{2} \doteq 1165.$$

*Exercise* 21.

Let $2x$ be the number of cattle: then $7x$ will be the number of sheep.

$$2x \times 168 + 7x \times 25 - 15\tfrac{1}{2} \doteq 6116\tfrac{1}{2}.$$
$$\therefore 511x \doteq 6132.$$

*Exercise* 22.

Let $x$ be the price of the bracelets in guineas. Then the brooch cost $7+\frac{x}{2}$.

$$x \doteq 7 + 7 + \frac{x}{2}.$$

*Exercise* 23.

Let $2x$ express the price of the black tea, per pound, in shillings: then $3x$ will express that of the green, and we shall have

$$2x + 3x = 2 \times (6\tfrac{1}{4}) = 12\tfrac{1}{2}.$$

*Exercise* 24.

$$x + y \doteq 133, \text{ and } \frac{x}{y} \doteq 18.$$

*Exercise* 25.

$$x - y \doteq 1, \text{ and } x^2 - y^2 \doteq 19.$$

Dividing the second equation by the first, we have

$$x + y = 19.$$

Take first the sum and then the difference of this and equation 1.

*Exercise* 26.

$$x + y \doteq 100, \text{ and } x^2 - y^2 \doteq 200.$$

Divide the second equation by the first, and then proceed as in Exercise 25.

*Exercise* 27.

$$x+\frac{y+z}{2}\doteq 46,\ldots\ldots\ldots\ldots(1).$$
$$y+\frac{x+z}{3}\doteq 30,\ldots\ldots\ldots\ldots(2).$$
$$z+\frac{x+y}{4}\doteq 29,\ldots\ldots\ldots\ldots(3).$$

Multiplying Eq. 2 by 3, and Eq. 3 by 4, we have

$$x+3y+\ z=\ 90,\ldots\ldots\ldots\ldots(4),$$
$$\text{and } x+\ y+4z=116,\ldots\ldots\ldots\ldots(5).$$

Subtract Eq. 1 from Eq. 4, and then from Eq. 5.

$$\frac{5y}{2}+\frac{z}{2}\doteq 44,\ldots\ldots\ldots\ldots(6).$$
$$\frac{y}{2}+\frac{7z}{2}\doteq 70,\ldots\ldots\ldots\ldots(7).$$

Multiply Eq. 6 by 2, and Eq. 7 by 10.

$$5y+\ z\doteq\ 88,\ldots\ldots\ldots\ldots(8),$$
$$5y+35z\doteq 700,\ldots\ldots\ldots\ldots(9).$$

Subtracting Eq. 8 from Eq. 9, we have

$$34z=612, \text{ and } \therefore z=18.$$

Substitute 18 for $z$ in Eq. 8. That gives $y=14$; and substituting 14 and 18 for $y$ and $z$ in Eq. 4 or 5, we have $x=30$.

*Exercise* 28.

Call the number in the second class $4\times 7x=28x$, in order that we may complete both proportions without fractional numbers. This we do by Pr. I of Ch. XII.

Then, number in 1st class $\doteq 3\times 7x=21x$,
and number in 2d class $\doteq 4\times 8x=32x$.

$$28x+21x+32x\doteq 81.$$
Hence $x\doteq 1$, $21x\doteq 21$, &c.

*Exercise* 29.

Let $3x$ and $5x$ stand for the respective numbers in the two boats at first. Then

$$3x + 6 \doteq 5x - 6.$$

*Exercise* 30.

Let $x$ denote the number of shillings spent by George. Then John has spent $2x + 5$.

George still possesses $105 - x$, and
John.................... $105 - 2x - 5 = 100 - 2x$.

$$\therefore 100 - 2x \doteq \frac{105 - x}{2} - 5.$$

*Exercise* 31.

Let $w$, $x$, $y$, and $z$ be the respective sums left to Edward, Richard, Alfred, and Harry, and $v$ the price of the house. Then we have

$$2w = v,\ x = 2v,\ 2y + z = v, \text{ and } z + \frac{x}{3} = v.$$

$$\text{Hence } w \doteq \frac{v}{2},\ x \doteq 2v,\ z \doteq v - \frac{2v}{3} = \frac{v}{3}, \text{ and } y \doteq v - \frac{v}{3} = \frac{2v}{3}.$$

$$\text{But } w + x + y + z \doteq v, \text{ or}$$

$$\frac{v}{2} + 2v + \frac{2v}{3} + \frac{v}{3} \doteq 8400.$$

*Exercise* 32.

Let $x$ represent the son's age seven years ago. Then $4x$ would be the father's at that time. The father's age *now* will be $4x + 7$. Seven years hence the father's age will be $4x + 14$, and the son's, $x + 14$.

$$\therefore 4x + 14 \doteq 2(x + 14) = 2x + 28.$$

$$\text{Hence } 2x \doteq 14, \text{ and } x = 7.$$

$$\therefore 4x + 7 \doteq 35.$$

*Exercise* 33.

Let $x$ stand for the number of oranges.

Then $\frac{4x}{5}$ will be the cost price of the whole, and $\frac{x}{3}+\frac{x}{2}$ will be the sum they were sold for, in pence.

$$\therefore \frac{x}{3}+\frac{x}{2} \doteq \frac{4x}{5}+6.$$

*Exercise* 34.

Let $x$ represent the boy's share, and consequently $2x$ the first man's. One third of the whole sum will be 27s. 6d., and consequently 30s. will be the second man's share. Then

$$x+2x+30 \doteq 82\tfrac{1}{2}.$$

*Exercise* 35.

Let $x$ be the sum the first draws at three sales, or the second at two. Then $\frac{x}{3}$ will be the price of one of the articles sold by the first, and $\frac{x}{2}$, of one of those sold by the second.

Let $3y$ denote the number of sales made by the second before they have drawn equal sums. The number made by the first in the same time will be $4y$; and the number from the beginning $4y+36$.

The sum drawn by the second will be $3y \times \frac{x}{2} = \frac{3}{2}y \times x$, and that drawn by the first will be $(4y+36) \times \frac{x}{3} = \frac{4y+36}{3} \times x$.

$$\therefore \frac{3}{2}y \times x \doteq \frac{4y+36}{3} \times x.$$

$$\therefore \frac{3}{2}y \doteq \frac{4y+36}{3},$$

$$\text{and } 9y \doteq 8y+72,$$

$$\therefore y \doteq 72, \text{ and } 3y \doteq 216.$$

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# CHAPTER XVI.

## QUADRATIC EQUATIONS.

### Problem I.

*The rule* requires no demonstration.

*Exercise* 3.

Multiplying by 3, we find

$$6x^2 - 4 - 6 = 18 - 4x^2 + 3x^2.$$
$$\therefore 7x^2 \doteq 28.$$
$$\text{Hence } x^2 \doteq 4, \text{ and } x = \pm 2.$$

*Exercise* 4.

Squaring both sides,

$$(x+1) \times (x-1) \doteq x^2 - 2 + \frac{1}{x^2}.$$
$$\therefore x^2 - 1 \doteq x^2 - 2 + \frac{1}{x^2}.$$
$$\text{Hence } 1 \doteq \frac{1}{x^2}, \text{ and } x^2 \doteq 1.$$

PROBLEM II.

In the rule, no principle is employed different from those already demonstrated.

*Exercise* 1.

Adding the square of 5 to each side,

$$x^2 + 10x + 25 \doteq 100.$$

Extracting the square root of each member,

$$x + 5 \doteq \pm 10.$$

*Exercise* 3.

Dividing by 5, we obtain

$$x^2 - 4x + 3 = 0.$$

Adding 1 to both sides,

$$x^2 - 4x + 4 \doteq 1.$$

Extracting the square root.

$$x - 2 \doteq \pm 1.$$

*Exercise* 8.

$$z^2 - 40z \doteq 10z + 400.$$
$$z^2 - 50z \doteq 400.$$
$$z^2 - 50z + 625 \doteq 1025.$$
$$z - 25 \doteq \pm \sqrt{1025}.$$

*Exercise* 11.

$$3x^2 - 2x \doteq 8.$$
$$x^2 - \tfrac{2}{3}x \doteq \tfrac{8}{3} = \tfrac{24}{9}.$$
$$x^2 - \tfrac{2}{3}x + \tfrac{1}{9} \doteq \tfrac{25}{9}.$$
$$x - \tfrac{1}{3} \doteq \pm \tfrac{5}{3}.$$
$$x \doteq \tfrac{1}{3} \pm \tfrac{5}{3} = \tfrac{6}{3} \text{ or } -\tfrac{4}{3}.$$

*Exercise* 12.

$$5x^2 - 6x \doteq 8.$$
$$x^2 - \tfrac{6}{5}x \doteq \tfrac{8}{5} = \tfrac{40}{25}.$$
$$x^2 - \tfrac{6}{5}x + \tfrac{9}{25} \doteq \tfrac{49}{25}.$$
$$x - \tfrac{3}{5} \doteq \pm \tfrac{7}{5}.$$

*Exercise* 13.

$$5y^2 - 4y \doteq 156.$$
$$y^2 - \tfrac{4}{5}y \doteq 31{\cdot}2.$$
$$y^2 - \tfrac{4}{5}y + ({\cdot}4)^2 \doteq 31{\cdot}2 + 0{\cdot}16 = 31{\cdot}36.$$
$$y - 0{\cdot}4 \doteq \pm 5{\cdot}6.$$

*Exercise* 14.

Subtract 4 from both members.

$$z^2 - 8z^* + 16 \doteq -4.$$

*Exercise* 15.

Multiply by 3.

$$x^2 - \tfrac{3}{5}x \doteq 216.$$
$$x^2 - \tfrac{3}{5}x + (\tfrac{3}{10})^2 \doteq 216\tfrac{9}{100}.$$
$$x - \tfrac{3}{10} \doteq \pm 14\tfrac{7}{10}.$$

*Exercise* 16.

Multiply by $x$.

$$125 - 3x - 4x^2 \doteq 2x.$$
$$4x^2 + 5x \doteq 125.$$
$$x^2 + \tfrac{5}{4}x \doteq \tfrac{125}{4} = \tfrac{2000}{64}.$$
$$x^2 + \tfrac{5}{4}x + (\tfrac{5}{8})^2 \doteq \tfrac{2025}{64}.$$
$$x + \tfrac{5}{8} \doteq \pm \tfrac{45}{8}.$$

* There is an error in the question, $+8z$ for $-8z$.

*Exercise* 17.

Multiply by $x^2$.

$$20x-10x-8-2x^2 \doteq 0.$$
$$2x^2-10x \doteq -8.$$
$$x^2-5x \doteq -4.$$
$$x^2-5x+\tfrac{25}{4} \doteq 2\tfrac{1}{4}.$$
$$x-\tfrac{5}{2} \doteq \pm\tfrac{3}{2}.$$

*Exercise* 18.

Multiplying by $y+3$.

$$2y^2+6y-8y+6 \doteq 9y+27.$$
$$y^2-\tfrac{11}{2}y \doteq \tfrac{21}{2}=\tfrac{168}{16}.$$
$$y^2-\tfrac{11}{2}y+\tfrac{121}{16} \doteq \tfrac{289}{16}.$$
$$y-\tfrac{11}{4} \doteq \pm\tfrac{17}{4}.$$
$$y \doteq \tfrac{11}{4}\pm\tfrac{17}{4}=+\tfrac{28}{4} \text{ or } -\tfrac{6}{4}.$$

*Exercise* 19.

Cubing both sides, $z^2-6z \doteq -8$.

*Exercise* 20.

Squaring both sides, we have

$$(x+4)\times(x-3)=(2x-6)^2.$$

That is, $x^2+x-12=4x^2-24x+36$.

$$3x^2-25x \doteq -48.$$
$$x^2-\tfrac{25}{3}x \doteq -16=-\tfrac{576}{36}.$$
$$x^2-\tfrac{25}{3}x+(\tfrac{5}{6})^2 \doteq \tfrac{49}{36}.$$
$$x-\tfrac{25}{6} \doteq \pm\tfrac{7}{6}.$$

*Exercise* 21.

Transposing $x$ we have

$$\sqrt{x}=15-x.$$

Squaring both sides,

$$x \doteq 225-30x+x^2.$$
$$\therefore x^2-31x \doteq -225=-\tfrac{900}{4}.$$
$$x^2-31x+(\tfrac{31}{2})^2 \doteq \tfrac{61}{4}.$$
$$x-\tfrac{31}{2} \doteq \pm\frac{\sqrt{61}}{2}.$$
$$x \doteq \frac{31\pm\sqrt{61}}{2}.$$

NOTE. Another method is pointed out in Note 2.

*Exercise* 22.

$$x^4+2x^2+1 \doteq 10201.$$
$$x^2+1 \doteq 101.$$

*Exercise* 23.

$$x^6-35x^3 \doteq -216 = -\tfrac{864}{4}.$$
$$x^6-35x^3+(\tfrac{35}{2})^2 \doteq \tfrac{361}{4}.$$

Extracting the square root,

$$x^3-\tfrac{35}{2} \doteq \pm\tfrac{19}{2}.$$
$$x^3 \doteq \frac{35 \pm 19}{2} = 27 \text{ or } 8.$$
$$\therefore x \doteq 3 \text{ or } 2.$$

*Exercise* 24.

$$y-\sqrt{y}+\tfrac{1}{4} \doteq 20\tfrac{1}{4} = \tfrac{81}{4}.$$
$$\sqrt{y}-\tfrac{1}{2} \doteq \pm\tfrac{9}{2}.$$
$$\sqrt{y} \doteq \frac{1 \pm 9}{2} = 5 \text{ or } -4.$$
$$y \doteq (+5)^2 \text{ or } (-4)^2.$$

NOTE. Although $y \doteq (+5)^2$ or $(-4)^2$, yet we cannot say that $y \doteq 25$ or $16$; for it is $=25$, only if we take the positive root of 25 for $\sqrt{x}$, and it is $=16$, only if we take the negative root of 16 for $\sqrt{x}$. The result is one of a very peculiar kind.

PROBLEM III.

*Exercise* 1.

Multiply the first equation by 2 and the second by 3.

$$6x^2+8y^2 \doteq 86.$$
$$6x^2-9y^2 \doteq 18.$$

Hence, $17y^2 \doteq 68$, $y^2 \doteq 4$, and $y \doteq \pm 2$.

Insert 4 for $y^2$ in Eq. 2, and we have

$$2x^2-12=6.$$

*Exercise* 2.

From the two proportions we obtain

$$x + y = 5x - 5y,$$
$$\text{and } xy = 96.$$

From the first of these two equations we find $y = \frac{2}{3}x$; and that value, substituted for $y$ in the second, gives

$$\tfrac{2}{3}x^2 = 96, \text{ or } x^2 = 144.$$

NOTE. When we find an answer in the form $x = \pm a$, and $y = \pm b$, it may either be meant that $x$ and $y$ may have either of these two values assigned to them respectively, without regard to each other; or it may be meant that the two values must be taken with corresponding signs,—in other words, that if the value of $x$ is taken with the upper sign, that of $y$ must be so also, and if the lower sign be used for the one, it must be used likewise for the other. The answer to Exercise 1 is given in the former sense, and that to Exercise 2 in the latter. If it be asked how the distinction arises, let it be observed that in Ex. 2 we found a certain relation to exist between $x$ and $y$, viz. $y = \frac{2}{3}x$, which could not be if $x$ and $y$ had contrary signs. In Ex. 1, on the other hand, no relation exists except between the *squares* of the two unknown quantities, and that relation is not affected by the sign given to either quantity, since the value of $x^2$ or $y^2$ is positive whatever be the sign of the value of $x$ or $y$. A new notation is wanted to show the distinction of the two significations in the case of a double answer.

*Exercise* 3.

Dividing the first equation by the second, we have $z^2 = \frac{9}{4}$, and $\therefore z = \frac{3}{2}$.

*Exercise* 4.

From Eq. 1 we obtain $x = 3y + 9$.

Substitute $3y + 9$ for $x$ in Eq. 2, giving

$$y^2 - 4y + 6y + 18 = 26,$$
$$\therefore y^2 + 2y = 8,$$
$$\text{and } y^2 + 2y + 1 = 9.$$

NOTE. The values found for $x$ and $y$ are not to be taken indiscriminately. When $x$ is 15, $y$ must be 2, and when $x$ is $-4$, $y$ must be $-1$. This follows from the relation established between the two in Eq. 1.

*Exercise* 5.

From Eq. 2, we have

$$5x = 8y - 12, \ldots\ldots\ldots\ldots (3),$$

$$\text{and } x = \frac{8y - 12}{5}, \ldots\ldots\ldots\ldots (4).$$

Multiplying these two equations together,

$$5x^2 = \frac{64y^2 - 192y + 144}{5}.$$

Substituting this value for $5x^2$ in Eq. 1,

$$\frac{24y^2 - 192y + 144}{5} = 72.$$

Divide by 24 and multiply by 5.

$$y^2 - 8y + 6 = 15.$$

From this we find $y = 9$ or $-1$, and these two values substituted successively for $y$ in Eq. 4, give $x = 12$, or $x = -4$.

NOTE. If we had substituted 9 and $-1$ successively for $y$ in Eq. 1 instead of Eq. 4, we should have found $x = \pm 12$, and $x = \pm 4$. But the values $-12$ and $+4$, although answering satisfactorily for $x$ in Eq. 1, yet fail when applied to Eq. 2. The reason is, that we have merely found, *first*, that 9 and $-1$ may either of them be substituted for $y$ in either Eq. 1 or Eq. 2; and, *secondly*, that if $y$ be 9, the value $+12$ or $-12$ for $x$ will satisfy Eq. 1; or if $y$ be $-1$, then the value $+4$ or $-4$ will also satisfy Eq. 1. But we have had no reason to suppose, from any thing that has been done, that *all* of these values will answer for $x$ in Eq. 2; and, in fact, they do not. All that we know, from the last-mentioned mode of operation, is, that if there is any value of $x$ that will satisfy *both* equations (in other words, if the data are possible), then that value must be either $+12$, $-12$, $+4$, $-4$. When these four values are *tried, then,* two of them are found to answer for Eq. 2, and

(since they all answer for Eq. 1), these two must be the true values of $x$, answering *all* the assigned conditions.

*Exercise* 6.

Square Eq. 1.

$$x^2 + 2xy + y^2 = 1156.$$

From this subtract four times Eq. 2.

$$x^2 - 2xy + y^2 = 64.$$

Extract the square root.

$$x - y = \pm 8 \ldots\ldots\ldots\ldots (3).$$

Proceed with Equations 1 and 3, as in Exercise 1 of Pr. II of Simple Equations.

NOTE. The student should be directed to pay particular attention to this solution, and to keep the method in memory, as the same or a similar method may often be employed to great advantage.

*Exercise* 7.

Take successively the sum and difference of Eq. 1 and twice Eq. 2, and we have

$$x^2 + 2xy + y^2 = 64,$$
$$\text{and } x^2 - 2xy + y^2 = 1.$$

Extract the square roots.

$$x + y = \pm 8, \text{ and } x - y = \pm 1.$$

With these proceed as in the preceding Exercise.

*Exercise* 8.

Double Eq. 1 and square Eq. 2.

$$2x^2 \ldots\ldots\ldots + 2y^2 = 17.$$
$$x^2 - 2xy + y^2 = 1.$$

Subtracting, we obtain

$$x^2 + 2xy + y^2 = 16.$$
$$\text{and } \therefore x + y = \pm 4.$$

Proceed as in the two preceding Exercises.

*Exercise* 9.

Double Eq. 1 and square Eq. 2.

$$2x \ldots\ldots\ldots\ldots + 2y \doteq 62{\cdot}5.$$
$$x + 2\sqrt{x}\sqrt{y} + y \doteq 56{\cdot}25.$$

Subtracting, we have

$$x - 2\sqrt{x}\sqrt{y} + y = 6{\cdot}25.$$

Extracting the square root,

$$\sqrt{x} - \sqrt{y} \doteq \pm 2{\cdot}5.$$

Take the sum and difference of this and Eq. 2,

$$2\sqrt{x} \doteq 10 \text{ or } 5, \text{ and } 2\sqrt{y} \doteq 5 \text{ or } 10.$$
$$4x \doteq 100 \text{ or } 25, \text{ and } 4y \doteq 25, \text{ or } 100.$$

*Exercise* 10.

From the given proportion and equation we obtain

$$3x + 3z = 11x - 11z, \ldots\ldots\ldots\ldots(3),$$
$$\text{and } xz - 3x - 5z + 15 = 45, \ldots\ldots\ldots\ldots(4).$$

$$\text{From Eq. 3, } z \doteq \tfrac{4}{7}x, \ldots\ldots\ldots\ldots(5).$$
$$\text{From Eq. 4, } xz - 3x - 5z \doteq 30, \ldots\ldots\ldots\ldots(6).$$

Substituting $\frac{4}{7}x$ for $z$ in Eq. 6,

$$\tfrac{4}{7}x^2 - 3x - \tfrac{20}{7}x \doteq 30.$$
$$\text{Hence } 4x^2 - 41x \doteq 210.$$

From this we obtain $x = 14$ or $-3\frac{3}{4}$, and substituting these values for $x$ in Eq. 3, we find $z$.

*Exercise* 11.

Cube Equation 2.

$$x^3 + 3x^2y + 3xy^2 + y^3 \doteq 125.$$

Subtract Eq. 1, and divide the remainder by 3.

$$x^2y + xy^2, \text{ or } (x + y) \times xy \doteq 30.$$

For $x + y$ substitute its equal 5.

$$5xy \doteq 30.$$
$$\therefore 4xy \doteq 24.$$

Subtract this from the square of Eq. 1.

$$x^2-2xy+y^2 \doteq 1.$$
$$\therefore x-y \doteq \pm 1.$$

*Exercise* 12.

Subtract Eq. 1 from Eq. 2.

$$y+2z \doteq 14, \text{ or } y \doteq -2z+14, \ldots\ldots\ldots\ldots (3).$$

Subtract Eq. 2. from twice Eq. 1.

$$x-z \doteq -2, \text{ or } x \doteq z-2, \ldots\ldots\ldots\ldots (4).$$

$$\text{From Eq. 3, } y^2 \doteq 4z^2-56z+196.$$

$$\text{From Eq. 4, } x^2 \doteq z^2-4z+4.$$
$$\therefore x^2+y^2+z^2 \doteq 6z^2-60z+200=50.$$
$$\text{Hence, } z^2-10z+25 \doteq 0,$$
$$\text{and } z-5 \doteq 0, \text{ or } z \doteq 5.$$

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# CHAPTER XVII.

## QUESTIONS PRODUCING QUADRATIC EQUATIONS.

*Exercise* 1.

Let the parts be $x$ and $10-x$. Then

$$2x^2-20x+100 \doteq 54\tfrac{1}{2}$$
$$\therefore x^2-10x \doteq -22\tfrac{3}{4},$$
$$\text{and } x^2-10x+25 \doteq 2\tfrac{1}{4}.$$

Or thus:—

Let the parts be $5+y$ and $5-y$. Then

$$50+2y^2 \doteq 54\tfrac{1}{2}.$$
$$2y^2 \doteq 4\tfrac{1}{2}.$$
$$y^2 \doteq 2\tfrac{1}{4}.$$
$$y=1\tfrac{1}{2}.$$
$$\therefore 5+y \doteq 6\tfrac{1}{2}, \text{ and } 5-y=3\tfrac{1}{2}.$$

NOTE. The latter process is the preferable one.

*Exercise* 2.

Let the parts be expressed by $50+x$ and $50-x$.

$$2500-x^2 \doteq 2211.$$
$$x^2 \doteq 289.$$
$$x \doteq 17.$$

*Exercise* 3.

Let $4x$ and $7x$ represent the two numbers. Then

$$65x^2 \doteq 4160.$$

*Exercise* 5.

Let the parts be $x$ and $21-x$. Then

$$x^2 : 441-42x+x^2 :: 9 : 16.$$
$$\therefore 16x^2 \doteq 3969-378x+9x^2$$
$$7x^2+378x \doteq 3969.$$

*Exercise* 6.

Let the parts be $50+x$ and $50-x$.

$$2500-x^2 \doteq 200x.$$
$$\therefore x^2+200x \doteq 2500.$$

Or let the parts be $y$ and $100-y$.

$$100y-y^2 \doteq 10{,}000-200y.$$
$$y^2-300y \doteq -10{,}000.$$

*Exercise* 7.

Let $x$ be the rate in miles per hour. Then $175 \div x$ is the number of hours taken to the journey. The supposed rate would have been $x+5$, and the time taken, $175 \div (x+5)$ hours.

$$\therefore \frac{175}{x} \doteq \frac{175}{x+5} + 1\tfrac{3}{4}.$$

Multiplying by $x \times (x+5)$, we have

$$175x+875 \doteq 175x+\tfrac{7}{4}x^2+8\tfrac{3}{4}x.$$
$$\therefore \tfrac{7}{4}x^2+8\tfrac{3}{4}x \doteq 875.$$
$$\text{Hence } 7x^2+35x \doteq 3500,$$
$$\text{and } x^2+5x \doteq 500.$$

*Exercise* 8.

Let $x$ be the greater, and $y$ the smaller.

$$\text{Then, } (x-y)\times x = 104, \text{ and } (x-y)\times y = 88.$$
$$\text{Or, } x^2 - xy = 104, \text{ and } xy - y^2 = 88.$$

Subtract the second equation from the first.

$$x^2 - 2xy + y^2 = 16.$$
$$\therefore x - y = \pm 4.$$

By this divide each of the two given equations.

*Exercise* 9.

Let $x$ be the number; and consequently $x+12$, the number of dozens, making the whole cost $\frac{x}{12}\times 28 = \frac{7x}{3}$ shillings. If 6 guineas, or 126 shillings, be the sum received for the whole, viz. $(x+12)$ dozens. Then, as the number of dozens to 1 dozen, so is the selling price of the whole to the selling price of 1 dozen. That is,

$$\text{As } \frac{x}{12} : 1 :: 126 : 126\times\frac{12}{x} = \frac{1512}{x},$$

which is the sum received for 1 dozen, or the clear profit on the whole. Consequently

$$\frac{7x}{3} + \frac{1512}{x} = 126.$$

Dividing by 7, we obtain

$$\frac{x}{3} + \frac{216}{x} = 18.$$
$$\therefore x^2 + 648 = 54x.$$

*Exercise* 10.

Call the smaller number $x$: the greater will be $x+4$. Then

$$x^2 + 4x = 117.$$

Or, call the greater $y+2$, and the smaller $y-2$. Then

$$y^2 - 4 = 117.$$

*Exercise* 11.

Let $x+1$ and $x-1$ be employed to denote the two numbers. Then $x^3+3x^2+3x+1$, and $x^3-3x^2+3x-1$ represent their cubes, the difference of which is

$$6x^2-2=152.$$

*Exercise* 12.

Let $x$ be the number of contributors. Then $120 \div x$ is the sum each would contribute, in shillings. If the other eight had joined, the number would have been $x+8$, and the sum paid by each $120 \div (x+8)$.

$$\therefore \frac{120}{x+8}+\frac{1}{2}=\frac{120}{x}.$$

Multiplying by $2(x+8)x$, and cancelling,

$$x^2+8x=1920.$$
$$\text{Hence } x=40.^*$$

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## CHAPTER XVIII.

## ARITHMETICAL PROGRESSIONS.

### THEOREM.

*Demonstration.*

If $a$ be put for the first term and $l$ for the last, the common difference being $d$, the progression will be expressed thus :—

$$a,\ a\pm d,\ a\pm 2d,\ \ldots\ldots\ldots\ldots l\mp 2d,\ l\mp d,\ l.$$

The same progression reversed, is

$$l,\ l\mp d,\ l\mp 2d \ldots\ldots\ldots\ldots a\pm 2d,\ a\pm d,\ a.$$

Adding together these two forms of the progression in the order of the terms, we have the sum of every pair $=a+l$; and, if the number of terms is odd, the middle

* In the text there is an erratum of 48 for 40.

term may be expressed in the one line by $a \pm md$, and in the other by $l \mp md$, the sum of the two being also $a + l$.

*Exercise* 1.

By the theorem the middle term will be $(7 + 19) \div 2 = 26 \div 2 = 13$. The term intermediate between 7 and 13 will be $(7 + 13) \div 2 = 20 \div 2 = 10$; and that intermediate to 13 and 19 will be $(13 + 19) \div 2 = 32 \div 2 = 16$. Therefore the progression stands

$$7,\ 10,\ 13,\ 16,\ 19.$$

NOTE. The same thing might be done more briefly by first finding the common difference $d = (19 - 7) \div 4$.

*Exercise* 2.

$$\text{Middle term} = \frac{156a + 246a + 10d}{2} = 201a + 5d.$$

PROBLEM I.

The *Rule* is derived immediately from the general form of a progression employed in illustrating the Definition, viz.—

$$a,\ a \pm d,\ a \pm 2d,\ a \pm 3d,\ \ldots\ldots\ldots\ldots a \pm (n-1)d,$$

in which we observe that the 2d term is $a \pm 1d$; the 3d, $a \pm 2d$; the 4th, $a \pm 3d$; and consequently, in general, the $n$th, $a \pm (n-1)d$.

The *Exercises* are too simple to require a solution here.

PROBLEM II.

*Demonstration of the Rule.*

Express the progression and reverse it, as in the preceding demonstration of the Theorem, putting also $s$ for the sum and $n$ for the number of terms. Then, on adding the two together, we find

$$2s = (a + l) + (a + l) + (a + l) \text{ to } n \text{ terms}$$
$$= (a + l) \times n.$$
$$\therefore s = \frac{(a + l) \times n}{2} \text{ or } \frac{a + l}{2} \times n.$$

*Exercise* 3.

The number of terms will evidently be 11.

$$\therefore s \doteq \frac{2a+10b}{2} \times 11 = 11a + 55b.$$

*Exercise* 4.

By Problem I, the 21st term will be $x + \frac{x}{2} \times 20 = 11x$.

$$\therefore s \doteq \frac{12x}{2} \times 21 = 126x.$$

*Exercise* 5.

By Pr. I, his last year's wage is $105 + 15 \times 39 = 690$sh. $=$ £34 : 10.

$$\therefore s \doteq \frac{795 \times 40}{2} = 795 \times 20\text{sh.} = £795.$$

# CHAPTER XIX.

## GEOMETRICAL PROGRESSIONS.

### DEFINITIONS.

*Exercise* 1.

In the first, $r \doteq 3$, and $n \doteq 6$.
In the second, $r \doteq \frac{1}{2}$, and $n \doteq 6$.

*Exercise* 2.

Ans. $7a$, $14a$, $28a$, $56a$, $112a$.

*Exercise* 3.

Ans. $4x$, $20xy$, $100xy^2$, $500xy^3$.

*Exercise* 4.

Since the second term is $y \div 3 = y \times \frac{1}{3}$, the common ratio is $\frac{1}{3}$. Therefore the last term but one must be $z \div \frac{1}{3} = z \times 3 = 3z$.

*Exercise* 5.

Since the second term is $\frac{x}{y}$, and the third $\frac{y}{x}$, the common ratio will be $\frac{y}{x} \div \frac{x}{y} = \frac{y}{x} \times \frac{y}{x} = \frac{y^2}{x^2}$.

$$\text{Then first term} \doteq \frac{x}{y} \div \frac{y^2}{x^2} = \frac{x}{y} \times \frac{x^2}{y^2} = \frac{x^3}{y^3}.$$

$$\text{and last term} \doteq \frac{y}{x} \times \frac{y^2}{x^2} = \frac{y^3}{x^3}.$$

NOTE. The statement in the Note is demonstrated thus :—

Let $p$ be any term, the succeeding term will be $pr$ and the next $pr^2$. Now $p : pr :: pr : pr^2$, for $p \div pr \doteq pr \div pr^2$.

Hence any term of a geometrical progression is the square root of the product of the two adjacent terms.

*Exercise* 6.

Ans. 3, 6, 12, 24, 48, 96,
and $a$, $3a$, $9a$, $27a$, $81a$.

THEOREM.

*Demonstration.*

If the first term be $a$ and the last $l$, the progression may be expressed thus :—

$$a,\ ar,\ ar^2,\ \ldots\ldots\ldots\ldots,\ \frac{l}{r^2},\ \frac{l}{r},\ l.$$

Then $a \times l \doteq al$, $ar \times \frac{l}{r} \doteq al$; $ar^2 \times \frac{l}{r^2} \doteq al$, and so on. If there is an odd number of terms, the middle term may be expressed as either $ar^m$ or $\frac{l}{r^m}$; and again, $ar^m \times \frac{l}{r^m} \doteq al$.

*Exercise* 1.

Last term $\times 13 \doteq 104^2$.
$\therefore$ last term $\doteq 104^2 \div 13 = 832$.

*Exercise* 2.

Since any part of a geometrical progression is also a geometrical progression, it follows from the Theorem that the square of any intermediate term is equal to the product of any two equidistant terms; and, consequently, that any intermediate term is the square root of the product of two terms equidistant from it on each side. Therefore, 3d term $\doteq \sqrt{(1\text{st} \times 5\text{th})} = \sqrt{(7a \times 567a^5b^4)} = \sqrt{3969a^6b^4} = 63a^3b^2$.

In like manner, 2d term $\doteq \sqrt{(1\text{st} \times 3\text{d})} = \sqrt{(7a \times 63a^3b^2)} = \sqrt{441a^4b^2} = 21a^2b$.

And, 4th term $\doteq \sqrt{(3\text{d} \times 5\text{th})} = \sqrt{(63a^3b^2 \times 567a^5b^4)} = \sqrt{35721a^8b^6} = 189a^4b^3$.

## Problem I.

The rule follows immediately from the definition of a geometrical progression. For, by that definition, if the first term be $a$, the common ratio $r$, and the number of terms $n$, the progression will be

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ldots\ldots,\ ar^{n-1},$$

in which any term is equal to the product of $a$ into that power of $r$ whose index is 1 less than the number of the term.

*Exercise* 1.

$$\text{Tenth term} \doteq 13 \times 2^9 = 13 \times 512 = 6656.$$

*Exercise* 2.

$$\text{Seventh term} \doteq \frac{27}{a^3} \times \left(\frac{a}{3}\right)^6 = \frac{3^3 \times a^6}{a^3 \times 3^6} = \frac{a^3}{3^3} = \frac{a^3}{27}.$$

## Problem II.

*Demonstration of the Rule.*

Expressing the progression in the general form shown in the Definitions, and putting $s$ for the sum of the terms,

$$s \doteq a + ar + ar^2 \ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1}, \ldots\ldots (1),$$
$$\therefore rs \doteq \qquad ar + ar^2 \ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n, \ (2),$$

every term in Eq. 1 being multiplied by $r$, but placed in Eq. 2 so as to be adjacent to the corresponding term in Eq. 1. That being done, let us subtract Eq. 1 from Eq. 2, and we have

$$rs - s, \text{ or } (r-1)s = ar^n - a.$$

$$\therefore s \doteq \frac{ar^n - a}{r-1} = \frac{ar^{n-1} - a}{r-1} + ar^{n-1},$$

which, expressed in words, is our rule.

*Exercise* 1.

By Pr. I, tenth term $\doteq 3 \times 2^9 = 3 \times 512 = 1536.$

$$\therefore \text{sum } \doteq \frac{1536-3}{2-1} + 1536 = 3069.$$

*Exercise* 2.

By Pr. I, eighth term $\doteq 413{,}343 \times (\frac{1}{3})^7 = 189.$

$$\frac{413{,}343 - 189}{1 - \frac{1}{3}} = 413{,}154 \times \tfrac{3}{2} = 619{,}731.$$

Sum $\doteq 619{,}731 + 189 = 619{,}920.$

*Exercise* 3.

$$r \doteq \tfrac{1}{2}, \quad 1 - \tfrac{1}{2} \doteq \tfrac{1}{2}.$$

Sum $\doteq (1-0) \div \frac{1}{2} + 0 = 1 \times \frac{2}{1} = 2.$

*Exercise* 4.

$$r \doteq \tfrac{2}{3}, \quad 1 - \tfrac{2}{3} \doteq \tfrac{1}{3}.$$

Sum $\doteq (1-0) \div \frac{1}{3} + 0 = 1 \times \frac{3}{1} = 3.$

---

# CHAPTER XX.

## UNRESOLVED EXERCISES.

### EXERCISES IN CHAPTER II, PROBLEM I.

Answers: (1), $203ab$; (2), $+239\sqrt{x}$; (3), $-2{,}133x^2$; (4), $-1670 \times \frac{y}{z}$; (5), $+12c$; (6), $-84d^2$; (7), $-232\sqrt{xy}$; (8), $+246az$; (9), $-16y^2$; (10), $-119abc$; (11), $+96a^2x^2$; (12), $+317\sqrt{z}$; (13), $63x + 58y + 159z$;

(14), $52a+29b$; (15), $+182\sqrt{ax^3}-1238\sqrt{bx^3}$; (16), $-117mx-19nx+116px$.

## Exercises in Chapter II, Problem II.

Answers: (1), $197x+28y$; (2), $-7a-216b$; (3), $-49\sqrt{x}-4\sqrt{y}$; (4), $12a^2+8ab-13b^2+20ac+4bc+39c^2-14ad+34bd+6cd+9d^2$; (5), $48(a^2-x^2)$; (6), $668(b+c)^3$; (7), $118\frac{ax+by}{xy}$; (8), $7(ab+cd)+55(ac+bd)$.

## Exercises in Chapter III.

Answers: (1), $+3362x^2y^2$; (2), $+66abc$; (3), $-66my$; (4), $-59\sqrt{x^5}$; (5), $+52\sqrt{ay}+8\sqrt{bz}$; (6), $-340mx^2-727ny^2$; (7), $+12a-11b+9c-13d$; (8), $-212x-222y-650z$; (9), $21(a+x)+23(a+y)$.

## Exercises in Chapter IV.

Answers: (1), $a-b$; (2), $-55w+47x-67y+68z$.

## Exercises in Chapter V, Problem II.

(1.) Answer, $35a^5x^2+12a^4x^3-48a^3x^4-92a^2x^5$.
(2.) ........., $-225{,}828a^2b^{10}y^8+131{,}364a^2b^8y^{10}$.

## Exercises in Chapter V, Problem III.

(1.) Answer, $21a^2-53ab+30b^2$.
(2.) ........., $1190c^2-2381cd+1190d^2$.
(3.) ........., $-49m^2+126mn-81n^2$.
(4.) ........., $a^4-2a^2b^2+b^4$.
(5.) ........., $a+27a^4$.
(6.) ........., $729x^6-243x^4y^2+27x^2y^4-y^6$.
(7.) ........., $4050(x-y)^2$.
(8.) ........., $+1118(y+z)^6$.
(9.) ........., $25(a+b)^2-49(a-b)^2$.

### Exercises in Chapter V, Problem IV.

(1.) Answer, $-420a^3b^3x^3y^3$.
(2.) ........., $a^3+a^2b+a^2c+abc+a^2d+abd+acd+bcd$.
(3.) ........., $x^4-14x^3+71x^2-154x+120$.
(4.) ........., $-x^4-y^4-z^4+2x^2y^2+2x^2z^2+2y^2z^2$.
(5.) ........., $a^6-117{,}649$.

### Exercises in Chapter VI, Problem I.

Answers: (1), $-12a^2b$; (2), $+\frac{8}{7}x^3y^2z$; (3), $-24\frac{y^2}{x^2}$.

### Exercises in Chapter VI, Problem II.

(1.) Answer, $5a^2-12ab-15b^2$.
(2.) ........., $-8\frac{m}{n}a^2+\frac{3}{4}mn$.

### Exercises in Chapter VI, Problem III.

(1.) Answer, $b+2y$.
(2.) ........., $2a-3b$.
(3.) ........., $-4x^2-3y^2$.
(4.) ........., $a^2+ab+b^2$.
(5.) ........., $5a^2+10a+20$.
(6.) ........., $z^4+5z^3+25z^2+125z+625$.
(7.) ........., $5a-3b+\frac{23b^2}{2a+11b}$.
(8.) ........., $5x^2+9z^2-\frac{9z^4}{5x^2-9z^2}$.
(9.) ........., $7a^2+10a-1+\frac{37a-76}{12a^2-3a+5}$.
(10.) ........., $2x-3y+\frac{9xy^2+3y^3}{4x^2+4xy+y^2}$.
(11.) ........., $6y-5+\frac{9y+30.}{5y^2-7y+6}$.
(12.) ........., $x^2+2x-3-\frac{8x^2+3x+12}{x^3-2x^2+3x-4}$.
(13.) ........., $a-2b-\frac{2a^2b^2+5ab^3-2b^4}{3a^3+5a^2b-2ab^2+b^3}$.

(14.) Answer, $2 + 12x + 36x^2 + 108x^3 + \&c.$
(15.) ........., $10 + 50y + 250y^2 + 1250y^3 + \&c.$

## Exercises in Chapter VII.

(1.) Answer, $216ab^2x^2$.
(2.) ........., $5x+4y$.
(3.) ........., $9a^2-2$.

*Exercise* 4.

$$
\begin{array}{l}
8a^4-6a^2+1^*)64a^6-48a^4+12a^2-2(8a^2 \\
\qquad\qquad\quad 64a^6-48a^4+\ \ 8a^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad 2)4a^2-2 \\
\qquad\qquad\qquad\qquad\qquad\quad 2a^2-1
\end{array}
$$

$$
\begin{array}{l}
2a^2-1)8a^4-6a^2+1(4a^2-1 \\
\qquad\quad 8a^4-4a^2 \\
\hline
\qquad\qquad -2a^2+1 \\
\qquad\qquad -2a^2+1
\end{array}
$$

$\therefore$ G. C. M. $\doteq 2a^2-1$.

*Exercise* 5.

$$
\begin{array}{l}
18x^3-53xy^2-28y^3) \\
\qquad 36x^4-48x^3y \ldots\ldots\ldots\ldots - 56xy^3 - 49y^4(2x \\
\qquad 36x^4 \ldots\ldots\ldots - 106x^2y^2 - 56xy^3 \\
\hline
\qquad\quad -48x^3y+106x^2y^2 \ldots\ldots\ldots\ldots - 49y^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\hline
\qquad\quad -144x^3y+318x^2y^2 \ldots\ldots\ldots\ldots -147y^4(-8y \\
\qquad\quad -144x^3y \ldots\ldots\ldots\ldots +424xy^3+224y^4 \\
\hline
\qquad\qquad 53y^2)+318x^2y^2-424xy^3-371y^4 \\
\hline
\qquad\qquad\qquad +6x^2-\ \ 8xy-\ \ 7y^2.
\end{array}
$$

$$
\begin{array}{l}
6x^2-8xy-7y^2)18x^3 \ldots\ldots\ldots -53xy^2-28y^3(3x+4y \\
\qquad\qquad\quad 18x^3-24x^2y-21xy^2 \\
\hline
\qquad\qquad\qquad +24x^2y-32xy^2-28y^3 \\
\qquad\qquad\qquad +24x^2y-32xy^2-28y^3
\end{array}
$$

$\therefore$ G. C. M. $\doteq 6x^2-8xy-7y^2$.

* *The first of the two given quantities, in this Exercise, should be $8a^4-6a^2+1$, instead of $16a^4-12a^2+2$.*

*Exercise* 6.

$$12a^5 + 32a^4b - 12a^3b^2 \doteq 4a^3 \times (3a^2 + 8ab - 3b^2),$$
$$\text{and } 8a^4 - 72a^2b^2 \doteq 8a^2 \times (a^2 - 9b^2).$$

Now the g. c. m. of $4a^3$ and $8a^2$ is $4a^2$, and the g. c. m. of $3a^2 + 8ab - 3b^2$ and $a^2 - 9b^2$, is $a + 3b$.

$\therefore$ the g. c. m. of the two given quantities is $4a^3 + 12a^2b$.

*Exercise* 7.

$$6y^4 - 10y^3 + 32y \doteq 2y \times (3y^3 - 5y^2 + 16), \text{ and}$$
$$10y^4 - 10y^3 - 20y^2 + 80y \doteq 10y \times (y^3 - y^2 - 2y + 8).$$

The g. c. m. of $2y$ and $10y$ is $2y$, and that of $3y^3 - 5y^2 + 16$ and $y^3 - y^2 - 2y + 8$, is $y^2 - 3y + 4$.

$\therefore$ the g. c. m. required is $2y^3 - 6y^2 + 8y$.

## Exercises in Chapter VIII, Problem I.

Answers: (1), 288; (2), $187a$; (3), $240x^2y^2$; (4), $15a^2b + 60ab^2$; (5), $16b^4 - 81c^4$.

## Exercises in Chapter VIII, Problem II.

Answers: (1), 168; (2), $x^3y^3z^3$; (3), 1680; (4), $1080a$; (5), $2310abcd$; (6), $81a^4 - 288a^2b^2 + 256b^4$.

## Exercises in Chapter IX, Problem I.

Answers: (1), $\dfrac{540z}{36}$; (2), $\dfrac{a^2 - x^2}{a+x}$; (3), $\dfrac{y^2 - 343}{y - 7}$.

## Exercises in Chapter IX, Problem II.

Answers: (1), $\dfrac{15az - 7x^2}{5z}$; (2), $\dfrac{61a^2 - 112ab + 3b^2}{8a}$;

(3), $\dfrac{11x^2 + 32xy + 11y^2}{6y}$; (4), $-\dfrac{162x^2}{7y + 9z}$;

(5), $\dfrac{0}{x^4 + 9x^2 + 81}$.

## Exercises in Chapter IX, Problem III.

(1.) Answer, $4y^3 - 5x^3$.

(2.) ........., $4a + 8b + \dfrac{56b^2}{6a - 12b}$.

(3.) ........., $4a^2 - 6ab + 9b^2$.

(4.) ........., $8y + 7z + \dfrac{77z^2}{6y - 11z}$.

(5.) ........., $3x - 5 + \dfrac{75x - 50}{8x^2 + 9x - 10}$.

(6.) ........., $a - 2b + 3c - \dfrac{12bc - 9c^2}{a + 2b - 3c}$.

## Exercises in Chapter IX, Problem IV.

(1.) Answer, $\dfrac{6mn^2y^2 - 10mn^2z^2}{2m^2n^2}$.

(2.) ........., $\dfrac{14a^3 + 21a^2b - 8ab^2 - 12b^3}{2ax + 3bx}$.

(3.) ........., $\dfrac{a^2 - x^2}{(a - x)^2}$.

## Exercises in Chapter IX, Problem V.

(1.) Answer, $\dfrac{24x^4}{25z^4}$, (G. c. m. $= 37$).

(2.) ........., $\dfrac{a^2 - ab}{a + b}$, (g. c. m. $= 4a + 5b$).

(3.) ........., $\dfrac{2y + 6}{2y - 6}$, (g. c. m. $= 4y^2 - 5y$).

(4.) ........., $\dfrac{4x + y}{3x^2 + 12xy}$, (g. c. m. $= 3x - 5y$).

## Exercises in Chapter IX, Problem VI.

(1.) Answer, $\dfrac{6a}{12}$, $\dfrac{8b}{12}$, and $\dfrac{9c}{12}$.

(2.) Answer, $\frac{8ay}{60x^2y^2}$, and $\frac{9ax}{60x^2y^2}$.

(3.) ........., $\frac{175n^2x}{420m^2n^2}$, $\frac{147mnx}{420m^2n^2}$, and $\frac{160m^2x}{420m^2n^2}$.

(4.) ........., $\frac{9ay+15by}{54xy}$, $\frac{16ax-20bx}{54xy}$, and $\frac{9a^2-6b^2}{54xy}$.

(5.) ........., $\frac{x^2+12x+36}{x^2-36}$, and $\frac{x^2-12x+36}{x^2-36}$.

(6.)* ........., Numerators, $a^4+20a^3+150a^2+250a+625$, $a^4-625$, and $a^4-20a^3+150a^2-250a+625$. Common denominator, $a^4-50a^2+625$.

## Exercises in Chapter IX, Problem VII.

(1.) Answer, $\frac{6x^2+2y^2}{x}$.

(2.) ........., $\frac{163c}{60}$.

(3.) ........., $\frac{27a^2+60b^2+64c^2}{72abc}$.

(4.) ........., $\frac{2a^2}{a^2-b^2}$.

### *Exercise* 5.

$$6a^2-8ab \doteq 2a \times (3a-4b), \text{ and}$$
$$15ab-20b^2 = 5b \times (3a-4b).$$

$\therefore$ the least common denominator is $2a \times 5b \times (3a-4b) = 30a^2b-40ab^2$, and the work will stand thus:—

$$\frac{2a^2-3b^2}{6a^2-8ab}(5b \doteq \frac{10a^2b-15b^3}{30a^2b-40ab^2}$$
$$\frac{2a^2+3b^2}{15ab-20b^2}(2a \doteq \frac{4a^3+6ab^2}{30a^2b-40ab^2}$$
$$\text{Sum} \doteq \frac{4a^3+10a^2b+6ab^2-15b^3}{30a^2b-40ab^2}.$$

* In the question, *a* should be substituted for *x*.

*Exercise* 6.

$$x^2+14x+49 \doteq (x+7)^2 \times (x+7),$$
$$x^2+49 \doteq (x+7) \times (x-7),$$
$$x^2-14x+49 \doteq (x-7) \times (x-7).$$

$\therefore$ least com. den. $\doteq (x+7)^2 \times (x-7)^2$
$= x^4 - 98x^2 + 2401.$

The multipliers are $(x-7)^2$, $x^2-49$, and $(x+7)^2$.

The numerators are,

$$x^4 - 28x^3 + 294x^2 - 1372x + 2401,$$
$$x^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - 2401,$$
$$x^4 + 28x^3 + 294x^2 + 1372x + 2401.$$

$$\text{Sum} \doteq 3x^4 \ldots\ldots + 588x^2 \ldots\ldots + 2401.$$

*Exercise* 7.

$$2a^2 + 2ax \doteq 2a \times (a+x), \text{ and}$$
$$2a^2 - 2ax \doteq 2a \times (a-x).$$

$\therefore$ l. c. d. $\doteq 2a \times (a+x) \times (a-x) \times (a^2+x^2) = 2a(a^4-x^4)$.

The multipliers are $(a-x)\ (a^2+x^2)$, $(a+x)\ (a^2+x^2)$, and $2a(a^2-x^2)$.

The numerators are

$$a^3 - a^2x + ax^2 - x^3,$$
$$a^3 + a^2x + ax^2 + x^3, \text{ and}$$
$$-2a^3 \ldots\ldots + 2ax^2 \ldots\ldots$$

$$\text{Sum} \doteq \ldots\ldots\ldots\ldots 4ax^2 \ldots\ldots$$

$$\text{Ans. } \frac{4ax^2}{2a(a^4-x^4)} = \frac{2x^2}{a^4-x^4}.$$

*Exercise* 8.

$$+\frac{a+b}{b}(a^2-b^2 \doteq \frac{a^3+a^2b-ab^2-b^3}{a^2b-b^3},$$
$$-\frac{a}{a+b}(ab-b^2 \doteq \frac{-a^2b+ab^2}{a^2b-b^3},$$
$$-\frac{a^3-a^2b}{a^2b-b^3}(1 \quad \doteq \frac{-a^3+a^2b}{a^2b-b^3}.$$

$$\text{Sum} \doteq \frac{a^2b-b^3}{a^2b-b^3} = \frac{1}{1} = 1.$$

## EXERCISES IN CHAPTER IX, PROBLEM VIII.

Answers: (1), $\frac{b}{30}$; (2), $\frac{45a^3+70a}{6x}$; (3), $\frac{56b^2}{25a^2-16b^2}$; (4), $\frac{12ax}{4a^2-9x^2}$.

## EXERCISES IN CHAPTER IX, PROBLEM IX.

Answers: (1), $\frac{x^4-2401}{x^4-81}$; (2), $\frac{27xyz}{35}$; (3), 1; (4), $\frac{16a^2}{21b^2}$; (5), $a^2+b^2$; (6), $13a^2+15b^2$; (7), $35a+18a-64a=-11a$; (8), $8x-9y+14z$; (9), $3(a^2-b^2)$; (10), $121x^2-\frac{4y^2}{9}$, or $\frac{1089x^2-4y^2}{9}$; (11), $a^2+\frac{ab}{c}-\frac{ac}{d}-\frac{b}{d}$, or $\frac{a^2cd+abd-ac^2-bc}{cd}$.

## EXERCISES IN CHAPTER IX, PROBLEM X.

Answers: (1), $\frac{14a^2xy}{15b}$; (2), $\frac{15(a+b)x^2}{31(a-b)y^2}$; (3), $\frac{1}{8x}-\frac{1}{9y}+\frac{1}{14z}$; (4), $\frac{x^2+2x-8}{x^2-2x-8}$; (5), $ab-a^2$; (6), $8x^2-\frac{2}{3}y$.

*Exercise* 4 is resolved thus:—

$$\frac{x^2+3x-10}{x^2-x-6}\times\frac{x^2+x-12}{x^2+x-20}=\frac{(x-2)(x+5)}{(x+2)(x-3)}\times\frac{(x-3)(x+4)}{(x+5)(x-4)}$$
$$=\frac{x-2}{x+2}\times\frac{x+4}{x-4}=\frac{x^2+2x-8}{x^2-2x-8}.$$

## EXERCISES IN CHAPTER X, PROBLEM I.

(1.) Answer, $-16807a^{15}b^{25}c^{35}$.
(2.) ........., $-1,000,000,000x^{54}y^{72}$.

### EXERCISES IN CHAPTER X, PROBLEM II.

(1.) Ans. $25a^2 - 60ab + 36b^2$.
(2.) …… $49x^2 - 112xy + 64y^2 + 126xz - 144yz + 81z^2$.
(3.) …… $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.
(4.) …… $81c^4 - 432c^3d + 864c^2d^2 - 768cd^3 + 256d^4$.
(6.) …… $36x - 60\sqrt{x}\sqrt{y} + 25y$.
(7.) …… $\frac{9x^2}{16} - 2xy + \frac{16y^2}{9}$.
(8.) …… $8a^3 - 6a^2b + \frac{3}{2}ab^2 - \frac{1}{8}b^3$.

*Exercise* 5.

The fifth power is an erratum, in the question, for the third power.

$(x-y-z)^2 \doteq x^2 - 2xy + y^2 - 2xz + 2yz + z^2$.
$(x-y-z)^3 \doteq x^3 - 3x^2y + 3xy^2 - y^3 - 3x^2z + 6xyz + 3xz^2 - 3y^2z - 3yz^2 - z^3$.

### EXERCISES IN CHAPTER X, PROBLEM III.

*Exercise* 1.

$$\left(\frac{12x-15y}{16x-10y}\right)^2 \doteq \frac{144x^2 - 360xy + 225y^2}{256x^2 - 320xy + 100y^2}.$$

*Exercise* 2.

$$\left(\frac{5+z}{3-z}\right)^4 \doteq \left(\frac{25+10z+z^2}{9-6z+z^2}\right)^2$$
$$= \frac{625 + 500z + 150z^2 + 20z^3 + z^4}{81 - 108z + 54z^2 - 12z^3 + z^4}.$$

### EXERCISES IN CHAPTER XI, PROBLEM I.

Answers: (1), $\pm 5{\cdot}2ab^2c^3$; (2), $-6xy^3z^4$; (3), $\pm 8c^3z^2$; (4), It has no square root; (5), $\frac{2ab^2}{3x}$.

### Exercises in Chapter XI, Problem II.

Answers: (1), $4b-3c$; (2),* $8ax-12ay$; (3), $7a-9b+11c$; (4),† $\frac{2a^3-3b^3}{a+b}$; (5), $\frac{2}{3}a-\frac{3}{4}b$.

### Exercises in Chapter XII, Problem I.

Answers: (1), $18ab^2y$; (2), $4ax+4x^2$; (3), $\frac{a}{b}$; (4), $9a^2-4b^2$; (5), $180x$; (6), 12 days.

*Exercise* 7.

In one minute

1st and 2d together fill $\frac{1}{70}=\frac{12}{840}$,
1st and 3d.............. $\frac{1}{105}=\frac{8}{840}$,
2d and 3d............... $\frac{1}{140}=\frac{6}{840}$.

Adding all these together, we find that twice the 1st, 2d, and 3d together would fill $\frac{26}{840}$ of the reservoir in 1 minute; or

1st, 2d, and 3d fill $\frac{13}{840}$.
But ..... 2d and 3d fill $\frac{6}{840}$.
∴ 1st..............fills $\frac{7}{840}=\frac{1}{120}$.

In like manner we find that the 2d alone fills $\frac{5}{840}=\frac{1}{168}$; and the 3d alone, $\frac{1}{840}$. Therefore the first requires 120 minutes, the second 168, and the third 840 minutes, when working alone.

### Exercises in Chapter XII, Problem II.

Answers: (1), 56; (2), $15ab$; (3), $6a^2+13ab+6b^2$.

### Exercises in Chapter XIV, Problem I.

Answers.

| | | |
|---|---|---|
| (1), $x \doteq 134$. | (3), $z \doteq 346$. | (5), $x \doteq 15\frac{1}{3}$. |
| (2), $x \doteq 46$. | (4), $y \doteq 9$. | (6), $x \doteq 208$. |

* 96 in the question ought to be 192.
† The co-efficient 6 in the numerator should be 12.

(7), $z \doteq 125$.
(8), $y \doteq 12\frac{1}{2}$.
(9), $x \doteq 11a + 25b$.
(10), $x \doteq 8$.
(11), $x \doteq 12$.
(12), $x \doteq 72\frac{12}{29}$.
(13), $x \doteq 24$.
(14), $x \doteq 90$.
(15), $y \doteq 13$.
(16), $x \doteq 1$.
(17), $y \doteq 2\frac{1}{2}$.
(18), $z \doteq 10\frac{1}{2}$.
(19), $z \doteq 5\frac{1}{3}$.
(20), $x \doteq 3$.
(21), $x \doteq 8$.
(22), $x \doteq 3$.
(23), $y \doteq 22$.
(24), $x \doteq 13$.
(25), $y \doteq 64$.
(26), $x \doteq 6$.
(27), $x \doteq 9$.
(28), $y \doteq 51$.
(29), $x \doteq 7$.
(30), $z \doteq 4\frac{1}{2}$.
(31), $x \doteq 11$.
(32), $x \doteq 21$.
(33), $x \doteq 5$.

Of the preceding only a few require detailed solutions, viz.—

*Exercise* 23.

Squaring both sides,

$$4 \times (5y + 11) \doteq 25y - 66.$$

*Exercise* 24.

Squaring both sides,

$$(x+5) \times (x-5) \doteq (x-1)^2,$$
$$\text{or, } x^2 - 25 \doteq x^2 - 2x + 1.$$

*Exercise* 25.

$$\sqrt{4y} \doteq 2\sqrt{y}, \text{ and } \sqrt{\frac{y}{4}} \doteq \frac{\sqrt{y}}{2} = \frac{1}{2}\sqrt{y}.$$
$$\therefore \tfrac{1}{2}\sqrt{y} + 12 \doteq 2\sqrt{y},$$
$$\text{and } \tfrac{3}{2}\sqrt{y} \doteq 12, \text{ or } \sqrt{y} \doteq 8.$$

*Exercise* 26.

Again squaring each side,

$$(3x+31)\,.\,(3x-9) \doteq (3x+3)^2.$$
$$\text{That is, } 9x^2 + 66x - 279 \doteq 9x^2 + 18x + 9.$$
$$\therefore 48x \doteq 288, \text{ and } x \doteq 6.$$

*Exercise* 27.

Dividing the numerator of the fraction by the denominator, we have

$$2\sqrt{x} - 5 = \sqrt{x} - 2.$$
$$\therefore \sqrt{x} \doteq 3.$$

*Exercise* 32.

Dividing the first and third terms by $x+7$,

$$x-7 : x :: 2 : 3.$$

## EXERCISES IN CHAPTER XIV, PROBLEM II.

Answers.

(1), $\begin{cases} x \doteq 9, \\ y \doteq 5. \end{cases}$

(2), $\begin{cases} x \doteq 12, \\ y \doteq 16. \end{cases}$

(3), $\begin{cases} y \doteq 8, \\ z \doteq 3. \end{cases}$

(4), $\begin{cases} x \doteq 2\frac{35}{38}, \\ z \doteq 1\frac{2}{19}. \end{cases}$

(5), $\begin{cases} x \doteq 22, \\ z \doteq 10. \end{cases}$

(6), $\begin{cases} x \doteq 2\frac{1}{4}, \\ y \doteq 1\frac{1}{3}. \end{cases}$

(7), $\begin{cases} x \doteq 5, \\ y \doteq 36. \end{cases}$

(8), $\begin{cases} x \doteq 42, \\ y \doteq 66. \end{cases}$

(9), $\begin{cases} y \doteq 3\frac{3}{4}, \\ z \doteq 1\frac{1}{2}. \end{cases}$

(10), $\begin{cases} x \doteq 12, \\ z \doteq 7. \end{cases}$

(11), $\begin{cases} x \doteq 8{\cdot}7, \\ y \doteq 4{\cdot}1. \end{cases}$

(12), $\begin{cases} x \doteq 7\frac{1}{2}, \\ y \doteq 4\frac{1}{2}. \end{cases}$

(13), $\begin{cases} x \doteq 148, \\ z \doteq 150. \end{cases}$

(14), $\begin{cases} x \doteq 54, \\ y \doteq 32. \end{cases}$

*Solution of Exercise* 10.

Dividing Eq. 1 by Eq. 2, we have

$$x-z=5.$$

*Exercise* 11.

$$\begin{aligned} 7x-9y &\doteq 24, \ldots\ldots\ldots\ldots(3). \\ 5x+5y &\doteq 64. \\ x+y &\doteq 12{\cdot}8. \\ 7x+7y &\doteq 89{\cdot}6, \ldots\ldots\ldots\ldots(4). \end{aligned}$$

*Exercise* 12.

Clearing the equations of fractions, and collecting the terms, we have

$$\begin{aligned} 5x+y=42, &\ldots\ldots\ldots\ldots(3), \\ \text{and } 5y=3x, &\ldots\ldots\ldots\ldots(4). \end{aligned}$$

Multiplying Eq. 3 by 5,

$$25x+5y \doteq 210.$$

In this substitute $3x$ for $5y$.

$$28x \doteq 210.$$

*Exercise* 13.

Clearing the equations from fractions, &c.

$$-3x + 10z \doteq 1056,$$
$$\text{and } -6x + 19z \doteq 1962.$$

*Exercise* 14.

Multiplying Eq. 1 by 90, &c.

$$83x - 61y \doteq 2530.$$

In the proportion, equalizing the products of the extremes and means, multiplying the result by 4, &c.

$$x - 2y \doteq -10.$$

## Exercises in Chapter XIV, Problem III.

*Exercise* 1.

Add the three given equations together, halve the sum, and from the result subtract each of the given equations in succession. We obtain,

$$x = 1\tfrac{1}{2},\ y = 3\tfrac{1}{2},\ z = 4\tfrac{1}{2}.$$

*Exercise* 2.

Subtract Eqs. 1 and 2 successively from Eq. 3.

$$y + 2z \doteq 9.$$
$$3y + 4z \doteq 23.$$

Hence we find the values of $y$ and $z$, and afterwards of $x$, from Eq. 1 or 2. The values found are

$$x = 12,\ y = 5,\ z = 2.$$

*Exercise* 3.

Multiplying Eq. 1 by 4 and Eq. 2 by 2,

$$2x + \tfrac{4}{3}y + z \doteq 40; \text{ and}$$
$$2x + 2y + 2z \doteq 60.$$

Subtract these separately from Eq. 3.

$$\tfrac{5}{3}y + 3z \doteq 60.$$
$$y + 2z \doteq 40.$$

From these we find $y$ and $z$, and then $x$, from Eq. 2, viz.

$$x = 10,\ y = 0,\ z = 20.$$

*Exercise* 4.

Add Eq. 1 to Eq. 2, and halve the sum.

$$4x - y = 19.$$

Subtracting Eq. 3 from 3 times Eq. 2, and dividing the remainder by 2, we have

$$x + 7y = 70.$$

Hence, $x = 7$, $y = 9$, and $z = 11$.

*Exercise* 5.

Clearing the equations of fractions, we have

$$+8x + 8y - 5z = 100, \ldots\ldots\ldots\ldots (4),$$
$$-3x + 2y + 2z = 45, \ldots\ldots\ldots\ldots (5),$$
$$+4x - 3y + 4z = 68, \ldots\ldots\ldots\ldots (6).$$

Subtracting Eq. 6 from twice Eq. 5,

$$-10x + 7y = 22.$$

Adding twice Eq. 4 to 5 times Eq. 5,

$$x + 26y = 425.$$

Hence we find $x = 9$, $y = 16$, and $z = 20$.

## Exercises in Chapter XV.

(1.) Answer, $66\frac{2}{3}$.
(2.) ........., 16 and 18.
(3.) ........., £442.
(4.) ........., *A*, £770; *B*, £380; *C*, £750; *D*, £500.
(5.) ........., 51 and 49.
(6.) ........., 7 and 5 acres.
(7.) ........., half way between.
(8.) ........., 21, 42, 15, and 22.
(9.) ........., 12 and 7.
(10.) ........., $7\frac{2}{3}$ and $3\frac{1}{2}$.
(11.) ........., 93·5, and 77·4.
(12.) ........., 7 and 17 shillings.
(13.) ........., £45, 15s.
(14.) ........., 4·56, 4·70, 6·70, and 6·84.

(15.) Answer, £45.
(16.) ........., 73 and 27.
(17.) ........., 48 apples and 42 pears.
(18.) ........., 121 and 81.
(19.) ........., 100 guineas.
(20.) ........., gun, 4 guineas; belt 4; flask 2; and grouse, 1 guinea each.
(21.) Answer, 4 miles.
(22.) ........., 16 and 24 acres.
(23.) ........., 84.
(24.) ........., 27,894.
(25.) ........., $3\frac{3}{5}$ hours, or 3h. 36 m.
(26.) ........., $214\frac{3}{17}$ miles.

The solutions of the more difficult are as follows:—

*Exercise* 6.

Let $x$ and $12-x$ represent the areas of the two fields. Then

$$43x-28(12-x)\doteq 411.$$

*Exercise* 7.

Let $x$ be the number of hours which the first took in walking from Walton: then the other took $(x+1)$ hours in walking from Middleton. Consequently the distances they respectively walk before meeting are $4x$ and $3\times(x+1)$.

$$\therefore 4x+3(x+1)\doteq 24.$$

Hence we find $x=3$, $4x=12$, and $3(x+1)=12$.

*Exercise* 10.

$$(x+y)-(x-y)\doteq 7,$$
$$\text{and } (x-y)-y=\tfrac{2}{3}.$$

*Exercise* 12.

Let $x$ and $x+\frac{1}{2}$ stand for the sums they respectively spend per annum. Then $12-x$ and $12-(x+\frac{1}{2})$, or $12-x-\frac{1}{2}$, are the sums saved yearly; and the sums saved in the whole time, by each, are $480-40x$ and $480-40x-20$.

$$\therefore 480-40x\doteq 2(480-40x-20)+6.$$

Subtract $480-40x$ from both sides.

$$0 \doteq 480-40x-40+6.$$
$$\therefore 40x \doteq 446, \text{ and } x = 11\tfrac{3}{20} = £11, 3s.$$
$$12-x \doteq 17\text{sh}.$$
$$12-(x+\tfrac{1}{2}) \doteq 7\text{sh}.$$

*Exercise* 13.

Let $x$ be the sum each began with. Then

$$x+15-\tfrac{3}{4} \doteq 2 \times (x-15-\tfrac{3}{4}).$$

*Exercise* 14.

Let $w$, $x$, $y$, and $z$, represent the quantities required. Then

$$w+x+y+z \doteq 22{\cdot}8,$$
$$w+z \doteq x+y,$$
$$w \doteq \tfrac{2}{3}z,$$
$$\text{and } y \doteq x+2.$$

Substituting, in the first equation, for $x+y$, its value $w+z$, we have

$$2w+2z = 22{\cdot}8.$$
$$\therefore w+z \doteq 11{\cdot}4.$$
$$\text{That is, } \tfrac{2}{3}z+z \doteq 11{\cdot}4.$$

*Exercise* 15.

Let $x$ be the number of yards in the carpet: then $8x$ is the price of the carpet, in shillings, taking the best quality, and $8x-20$ is the money the lady had with her.

Next, $7\tfrac{1}{2}x$ will be the price of the carpet of the second quality, and $7\tfrac{1}{2}x+40$ is again the money in hand.

$$\therefore 8x-20 \doteq 7\tfrac{1}{2}x+40.$$

*Exercise* 16.

Let $x$ be the second part and $100-x$ the first: then the excess of the first above 49 will be $51-x$, and that of 49 above the second will be $49-x$. Then

$$51-x : 49-x :: 12 : 11.$$
$$\therefore 561-11x \doteq 588-12x.$$
$$\therefore x \doteq 27.$$

*Exercise* 17.

Let $x$ be the number of apples he bought, and $90-x$ the number of pears. Then $x \div 3$ will be the price of all the apples, in pence, and $\frac{2}{7} \times (90-x)$ that of the pears.

$$\therefore \frac{x}{3}+\frac{2}{7} \times (90-x) \doteq 28.$$

*Exercise* 18.

$$x-y \doteq 40, \ldots\ldots\ldots\ldots\ldots (1).$$
$$\sqrt{x}-\sqrt{y} \doteq 2, \ldots\ldots\ldots\ldots (2).$$

Divide Eq. 1 by Eq. 2, and we have

$$\sqrt{x}+\sqrt{y}=20, \ldots\ldots\ldots\ldots (3).$$

Taking the sum and difference of this and Eq. 2, we find $\sqrt{x}=11$, and $\sqrt{y}=9$.

*Exercise* 19.

Let $x$ represent Mr Johnson's subscription, in pounds. Mr Thomson's will be $3 \times (x-50)=3x-150$. Mr Wilson's, $4 \times [(3x-150)-50]=4 \times (3x-200)=12x-800$. Then

$$x+(3x-150)+(12x-800) \doteq 730.$$

Hence we find $x=£105$.

*Exercise* 20.

Let $w$ guineas be the value he attached to a brace of game; $x$, to the gun; $y$, to the shot-belt; and $z$, to the powder-flask. Then

$$x+y+z \doteq 5w.$$
$$x+y \doteq 4w.$$
$$y+z \doteq 3w.$$
$$y+z+6 \doteq 6w.$$

*Exercise* 21.

Let $2x$ represent the distance of the two places, in miles. Then the number of hours taken by the first will be $\frac{2x}{3}$; and by the second $\frac{x}{2}+\frac{x}{4}$.

$$\therefore \frac{x}{2}+\frac{x}{4} \doteq \frac{2x}{3}+\frac{1}{6}.$$

### *Exercise* 22.

Putting $x$ for the size of the one field in acres, and consequently $40-x$ for that of the other, the prices of the two will be $x^2$ and $(40-x)^2$.

$$\therefore x^2 : (40-x)^2 :: 4 : 9.$$

Extract the square root of all the terms, according to Theorem VII of Chapter XII.

$$x : 40-x :: 2 : 3.$$

### *Exercise* 23.

The digits being $x$ and $y$, the number will be $10x+y$, and the number reversed $10y+x$.

$$\therefore (10x+y)-(10y+x) \doteq \frac{10x+y}{2} - 6, \ldots\ldots\ldots\ldots(1),$$

$$\text{and } (10x+y)-(10y+x) \doteq \tfrac{3}{4}(10y+x), \ldots\ldots\ldots\ldots(2).$$

From Eq. 1 subtract $\frac{10x+y,}{2}$ and we have

$$\frac{10x+y}{2} - 10y - x = 6.$$

Doubling this, and collecting like terms,

$$8x-19y \doteq 12, \ldots\ldots\ldots\ldots(3).$$

Adding $10y+x$ to both sides of Eq. 2,

$$10x+y \doteq \tfrac{7}{4}(10y+x).$$

Multiplying by 4, &c., we find

$$x=2y, \ldots\ldots\ldots\ldots(4).$$

### *Exercise* 24.

Let $x$ denote the number of slain; $2x$, of the wounded; $y$, of the prisoners; $3y$, of those remaining fit for service; and therefore $x+2x+y+3y$, or $3x+4y$, of the whole army. But, by the conditions of the question,

$$3y \doteq \frac{3x+4y}{2} + 714,$$

$$\text{and } 2x-y \doteq 677.$$

From these two equations we find $y=4887$ and $x=2782$.

$$\therefore 3x+4y \doteq 27{,}894.$$

*Exercise* 25.

By the time the second train has got to the 30 mile station, the other must be 18 miles a-head of it, since $5:3::30:18$.

Let $x$ be the number of miles the first would travel *after that*, before the second overtook it: then $x+18$ will be the distance travelled by the second in the same time, or the distance from the 30 mile station to the point at which the one train overtakes the other.

$$\therefore x+18 : x :: 6 : 5.$$

Hence we find $x=90$, and $x+18=108$.

Next let $15y$ represent the uniform rate of travelling of the first, in miles per hour: then the rate at which the second commenced, is $25y$; and its rate afterwards $18y$.

$$\therefore 25y-18y \doteq 14.$$

Hence we obtain $y=2$, and $15y=30$.

Then $108 \div 30 \doteq 3\frac{18}{30}$, which is the time the first train would take after leaving the advanced station, or the time the second would take altogether.

*Exercise* 26.

Let $x$ be the distance from London, that from Edinburgh being $389-x$. Then the time taken from London to the place of meeting will be $\frac{100}{10}+\frac{x-100}{9}$; and that from Edinburgh will be $\frac{100}{7\frac{1}{2}}+\frac{389-x-100}{8}=13\frac{1}{3}+\frac{289-x}{8}$.

$$\therefore 10+\frac{x-100}{9} \doteq 13\tfrac{1}{3}+\frac{289-x}{8}.$$

Hence we find $x=214\frac{3}{17}$.

## Exercises in Chapter XVI, Problem I.

(1.) Answer, $x \doteq \pm 5$.
(2.) ........., $x \doteq \pm 14$.
(3.) ........., $x \doteq 7$ or $5$.

(4.) Answer, $y \doteq 15$ or $1\frac{2}{3}$.
(5.) ........., $x \doteq 1\frac{2}{5}$.

*Solution of Exercise* 3.

Multiplying by 12, we have

$$7x^2 - 84x + 252 = 7.$$

Dividing by 7,

$$x^2 - 12x + 36 \doteq 1.$$

Extracting the square root of each member,

$$x - 6 \doteq \pm 1.$$

*Exercise* 4.

Taking the products of the extremes and means,

$$16y^2 \doteq (5y - 15)^2.$$

Extract the square root of each member.

*Exercise* 5.

Multiplying by $5x^2$, and arranging the terms,

$$25x^2 - 70x + 49 \doteq 0.$$

Extracting the square root,

$$5x - 7 \doteq 0.$$

## Exercises in Chapter XVI, Problem II.

Answers.

(1.) $x \doteq 8$ or $-18$.
(2.) $y \doteq 9$ or $6$.
(3.) $x \doteq 11$ or $-5$.
(4.) $z \doteq \frac{2}{3}$ or $2\frac{1}{3}$.
(5.) $x \doteq 15$ or $7\frac{4}{5}$.
(6.) $x \doteq 4$ or $-3$.
(7.) $y \doteq 13$ or $1$.
(8.) $x \doteq 3$ or $\frac{1}{4}$.
(9.) $x \doteq 6$ or $\frac{1}{3}$.
(10.) $z \doteq 22$ or $-\frac{100}{37}$.
(11.) $x \doteq 15$ or $\frac{23}{34}$.
(12.) $x \doteq \pm\frac{1}{2}$ or $\pm\sqrt{\frac{1}{2}}$.
(13.) $x \doteq 3$ or $-\sqrt[3]{3}$.
(14.) $x \doteq \pm 2$ or $\pm\sqrt{(-4)}$,

or

$\pm\sqrt{(-6)}$ or $\pm\sqrt{(-\sqrt{-6})}$.

(15.) $x \doteq (+15)^2$ or $(-3)^2$.
(16.) $x \doteq \frac{1}{2}(3 \pm \sqrt{5})$.

The solutions of the more difficult are as follows:—

*Exercise* 5.

Multiply by 9.

$$x^2 - \tfrac{36}{5}x \doteq 117 = \tfrac{2925}{25}.$$
$$\therefore x^2 - \tfrac{36}{5}x + \tfrac{324}{25} \doteq \tfrac{3249}{25}.$$
$$\text{Hence } x - \tfrac{18}{5} \doteq \pm \tfrac{57}{5}.$$

*Exercise* 6.

Clearing the given equation of fractions, we have

$$15x^2 - 15x - 180 = 0,$$
$$\text{or } x^2 - x = 12.$$

*Exercise* 7.

Taking the products of the means and extremes,

$$12y^2 - 66y + 90 \doteq 6y^2 + 18y + 12.$$

*Exercise* 8.

Multiplying by $6x - 9$, we obtain

$$-6x^2 + 21x - 3 = 2x^2 - 5x + 3.$$
$$\therefore x^2 - \tfrac{26}{8}x \doteq -\tfrac{3}{4} = -\tfrac{48}{64}.$$
$$\text{and } x^2 - \tfrac{26}{8}x + \tfrac{169}{64} \doteq \tfrac{121}{64}.$$

*Exercise* 9.

Multiplying by $2x - 2$, and collecting like terms,

$$-21x^2 + 133x \doteq 42.$$

*Exercise* 10.

Clearing the given equation of fractions, &c.,

$$37z^2 - 714z \doteq 2200.$$

Dividing by 37 and completing the square,

$$z^2 - \tfrac{714}{37}z + \tfrac{127449}{1369} \doteq \tfrac{208849}{1369}.$$
$$\therefore z - \tfrac{357}{37} \doteq \pm \tfrac{457}{37}.$$

*Exercise* 11.

Multiply by $45x - 90$.

$$34x^2 - 533x + 345 \doteq 0.$$

Dividing by 34 and then completing the square,

$$x^2 - \tfrac{533}{34}x + (\tfrac{533}{68})^2 \doteq \tfrac{237169}{4624}.$$
$$\therefore x - \tfrac{533}{68} \doteq \pm \tfrac{487}{68}.$$

*Exercise* 12.

Dividing by 16 and transposing,

$$x^4 - \tfrac{3}{4}x^2 \doteq -\tfrac{1}{8} = -\tfrac{8}{64}.$$

Completing the square,

$$x^4 - \tfrac{3}{4}x^2 + \tfrac{9}{64} \doteq \tfrac{1}{64}.$$

Hence we find $x^2 = \frac{1}{4}$ or $\frac{1}{2}$, and $x = \pm\frac{1}{2}$ or $\pm\sqrt{\frac{1}{2}}$.

*Exercise* 13.

Dividing by 5 and transposing,

$$x^6 - 24x^3 \doteq 81.$$

Completing the square, &c., we find

$$x^3 = 27 \text{ or } -3, \text{ and } x = 3 \text{ or } -\sqrt[3]{3}.$$

*Exercise* 14.

Completing the square, and extracting the square root,

$$x^4 - 5 \doteq \pm 11.$$
$$\therefore x^4 \doteq 16 \text{ or } -6, \text{ and } x^2 \doteq \pm 4 \text{ or } \pm\sqrt{(-6)}.$$

Hence $x \doteq \pm 2$ or $\pm\sqrt{(-4)}$, or $\pm\sqrt[4]{(-6)}$ or $\pm\sqrt{(-\sqrt{-6})}$.

*Exercise* 15.

Adding 81 to both sides to complete the square,

$$x - 12\sqrt{x} + 36 \doteq 81.$$

Extracting the square root,

$$\sqrt{x} - 6 \doteq \pm 9.$$
$$\therefore \sqrt{x} = 15 \text{ or } -3, \text{ and } x \doteq (+15)^2 \text{ or } (-3)^2.$$

NOTE. We cannot say, however, that $x \doteq 225$ or 9. See the Note to Ex. 24 of Pr. II, Chap. XVI.

*Exercise* 16.

$$x - \sqrt{x} + \tfrac{1}{4} = \tfrac{5}{4}.$$
$$\therefore \sqrt{x} \doteq \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} = \tfrac{1}{2}(1 \pm \sqrt{5}).$$
$$\text{and } x \doteq \tfrac{1}{2}(3 \pm \sqrt{5}).$$

## EXERCISES IN CHAPTER XVI, PROBLEM III.

### Answers.

(1.) $\begin{cases} x \doteq \pm 5, \\ y = \pm 6. \end{cases}$ (6.) $\begin{cases} x \doteq \pm 3\frac{1}{3}, \\ y = \pm 2\frac{1}{3}. \end{cases}$

(2.) $\begin{cases} x \doteq \pm 15, \\ y = \pm 12. \end{cases}$ (7.) $\begin{cases} x \doteq \pm 19, \\ y = \pm 1. \end{cases}$

(3.) $\begin{cases} x \doteq \pm \frac{2}{3} \text{ or } \pm \frac{3}{4}, \\ y = \pm \frac{3}{4} \text{ or } \pm \frac{2}{3}. \end{cases}$ (8.) $\begin{cases} x \doteq \pm 36, \\ y = \pm 55. \end{cases}$

(4.) $\begin{cases} x \doteq \pm 7\frac{1}{3}, \\ y = \pm 2\frac{1}{3}. \end{cases}$ (9.) $\begin{cases} x \doteq \pm 1\frac{1}{4}, \\ y = \pm \frac{4}{5}. \end{cases}$

(5.) $\begin{cases} y \doteq \pm 12. \\ z = \pm 4. \end{cases}$ (10.) $\begin{cases} x \doteq 10, \\ y = 12. \end{cases}$

(11.) $\begin{cases} x \doteq 15 \text{ or } -10{\cdot}92, \\ z = 7 \text{ or } -12{\cdot}44. \end{cases}$

(12.) $\begin{cases} y \doteq 7, \text{ or } -6, \text{ or } \frac{1}{2}(+1 \pm \sqrt{-231}). \\ z = 6, \text{ or } -7, \text{ or } \frac{1}{2}(-1 \pm \sqrt{-231}). \end{cases}$

*Solution of Exercise* 1.

Multiplying Eq. 1 by 3, and Eq. 2 by 4, we have

$$24x^2 - 15y^2 = 60,$$
$$\text{and } 24x^2 - 12y^2 = 168.$$
$$\therefore 3y^2 \doteq 108, \ y^2 = 36, \text{ and } y = \pm 6.$$

*Exercise* 2.

From the given proportion we obtain $4x = 5y$.
From the given equation, $x \times 5y = 900$.

$$\therefore x \times 4x, \text{ or } 4x^2 \doteq 900.$$

*Exercise* 3.

From Eq. 2 we have $2xy = 1$.
Taking the sum and difference of this and Eq. 1,

$$x^2 + 2xy + y^2 \doteq 2\tfrac{1}{144} = \tfrac{289}{144},$$
$$\text{and } x^2 - 2xy + y^2 \doteq \tfrac{1}{144}.$$
$$\therefore x + y \doteq \pm \tfrac{17}{12},$$
$$\text{and } x - y \doteq \pm \tfrac{1}{12}.$$

Taking half the sum and half the difference of these two equations we obtain the different values of $x$ and $y$. By *taking the* upper signs only in both equations we find

$x = +\frac{3}{4}$ and $y = +\frac{2}{3}$. By the lower signs only, we obtain $x = -\frac{3}{4}$ and $y = -\frac{2}{3}$. By taking the upper sign in the first equation with the lower in the second, we have $x = +\frac{2}{3}$ and $y = +\frac{3}{4}$; and by taking the lower sign in the first and the upper in the second, $x = -\frac{2}{3}$ and $y = -\frac{3}{4}$.

*Exercise* 4.

$$15x \doteq 22 \times (x-y), \text{ and}$$
$$x(x-y) \doteq 36\tfrac{2}{3} = \tfrac{110}{3}.$$
$$\therefore x-y \doteq \frac{110}{3x}, \text{ and}$$
$$15x \doteq 22 \times \frac{110}{3x}, \text{ \&c.}$$

*Exercise* 5.

Extracting the sq. roots of the terms of the proportion,

$$y-2z : y+2z :: 1 : 5.$$
$$\therefore y \doteq 3z,$$
$$\text{and } 3z^2 = 48, \text{ \&c.}$$

*Exercise* 6.

$$20x^2 - 28y^2 \doteq 69\tfrac{7}{9}.$$
$$20x^2 - 30y^2 \doteq 58\tfrac{8}{9}.$$
$$\therefore 2y^2 \doteq 10\tfrac{8}{9}, \text{ and } y^2 = 5\tfrac{4}{9} = \tfrac{49}{9}.$$

*Exercise* 7.

Extracting the square root of each member of Eq. 1,

$$x - 3y \doteq \pm 16.$$

By this divide Eq. 2,

$$x + 3y \doteq \pm 22.$$

*Exercise* 8.

Adding together the two given equations,

$$x^2 + 2xy + y^2 \doteq 8281.$$
$$\therefore x + y \doteq \pm 91.$$

By this dividing Eqs. 1 and 2 successively we obtain the values of $x$ and $y$.

*Exercise* 9.

From Eq. 1 subtract Eq. 2.

$$x^2 - 2xy + y^2 \doteq \tfrac{81}{400}.$$
$$\therefore x - y \doteq \pm \tfrac{9}{20}.$$

By this divide Eqs. 1 and 2 successively.

*Exercise* 10.

From Eq. 1, $y \doteq 22 - x$,......................(3).
From Eq. 2, $5xy - 5x + 5y \doteq 610$,............(4).

Substituting in Eq. 2 the value of $y$ found in Eq. 1, and collecting like terms, &c.,

$$5x^2 - 100x \doteq -500.$$

*Exercise* 11.

From Eq. 2, $9x^2 \doteq -9z^2 \ldots\ldots + 2466$.
From Eq. 1, $9x^2 \doteq 16z^2 + 136z + 289$.

Subtracting the first of these equations from the second,

$$0 \doteq 25z^2 + 136z - 2177.$$
$$\therefore z^2 + \tfrac{136}{25}z + \left(\tfrac{68}{25}\right)^2 \doteq \tfrac{59049}{625},$$
$$\text{and } z \doteq -\tfrac{68}{25} \pm \tfrac{243}{25}.$$

*Exercise* 12.

Square Eq. 1.

$$y^2 - 2yz + z^2 \doteq 1, \ldots\ldots\ldots\ldots (3).$$

In Eq. 2 substitute $x$ for $yz$.

$$x^2 + 16x \doteq 2436.$$
$$\therefore x^2 + 16x + 64 \doteq 2500,$$
$$\text{and } x + 8 \doteq \pm 50.$$
$$\therefore x = yz \doteq 42 \text{ or } -58, \ldots\ldots\ldots\ldots (4).$$

Add four times Eq. 4 to Eq. 3.

$$y^2 + 2yz + z^2 = 169 \text{ or } -231.$$
$$\therefore y + z \doteq \pm 13 \text{ or } \pm \sqrt{(-231)}.$$

## EXERCISES IN CHAPTER XVII.

### Answers.

(1.) $+15$ or $-5$.
(2.) $1\frac{1}{2}$ or $\frac{1}{2}$.
(3.) $50 \times (3 \pm \sqrt{5})$, and
$50 \times (-1 \mp \sqrt{5})$.
(4.) 75.
(5.) 28 years.
(6.) 7 and 12, or $-12$ and $-7$.
(7.) 35 and 25.
(8.) 13 and 15.
(9.) 9, 15, and 25.
(10.) 45.
(11.) 1849.
(12.) 80, 90, and 96.

*Solution of Exercise* 3.

Let $x$ be the larger part, and consequently $100-x$ the smaller. Then,

$$100 \times (100-x) \doteq x^2.$$
$$\therefore x^2 + 100x \doteq 10{,}000,$$
$$\text{and } x^2 + 100x + 2{,}500 \doteq 12{,}500 = 2{,}500 \times 5.$$
$$\therefore x + 50 \doteq \pm 50\sqrt{5}.$$

*Exercise* 4.

Let $x$ represent the number bought, and $\therefore x-10$ the number sold. Then $1350 \div x$ will be the cost price of each horse, and $1306\frac{1}{2} \div (x-10)$ will be the selling price of each.

$$\therefore \frac{1306\frac{1}{2}}{x-10} \doteq \frac{1350}{x} + 2\tfrac{1}{10}.$$
$$\text{Hence, } 7x^2 + 75x = 45{,}000.$$

*Exercise* 5.

Let $x$ be the years the first has been in business, and $x-3$ the second. Then

$$\frac{6300}{x} \doteq \frac{6300}{x-3} - 27.$$
$$\therefore x^2 - 3x \doteq 700.$$

*Exercise* 6.

Let $x$ and $y$ be the two numbers, $x$ being the greater: then

$$(y+5) \times (x-5) \doteq 84.$$
$$(y+3) \times (x-3) \doteq 90.$$
$$\therefore xy + 5x - 5y \doteq 109, \ldots\ldots\ldots\ldots(1),$$
$$\text{and } xy + 3x - 3y \doteq 99, \ldots\ldots\ldots\ldots(2).$$

Subtracting and dividing the remainder by 2,

$$x - y \doteq 5, \text{ or } x \doteq y + 5.$$

Substituting $y + 5$ for $x$ in Eq. 1, we have

$$y^2 + 5y + 5y + 25 - 5y = 109,$$
$$\text{or } y^2 + 5y = 84.$$

*Exercise* 7.

$$\frac{350}{x} \doteq \frac{350}{x + 10} + 4.$$

*Exercise* 8.

Let $x$ be the price of the first piece per yard, and therefore $195 \div x$ the number of yards of it. Then $195 \div x$ will be the price per yard of the second piece, and $x$ the number of yards of it.

$$\text{Then } x + \frac{195}{x} \doteq 28.$$

*Exercise* 9.

Let $x$, $y$, and $z$ represent the parts in the order of their magnitudes. Then

$$x + y + z \doteq 49,^* \ldots\ldots\ldots\ldots\ldots(1),$$
$$(x - y) - (y - z) \doteq 4, \ldots\ldots\ldots\ldots(2),$$
$$\text{and } xz \doteq 225, \ldots\ldots\ldots\ldots\ldots\ldots(3).$$
$$\text{From Eq. 2, } x - 2y + z \doteq 4.$$

Subtracting this from Eq. 1, we have

$$3y = 45, \text{ or } y = 15.$$
$$\text{From this and Eq. 1, } x + z \doteq 34.$$

From the square of the last equation subtract 4 times Eq. 3, extract the square root, &c.

*Exercise* 10.

$$\frac{990}{x + 54} \doteq \frac{990}{x} - 12.$$

* Correction for 39.

Let $x$ be put for the number of scores, and consequently $20x + 9$ for the number of sheep. Then,

$$\frac{x}{2} - \sqrt{(20x + 9)} \doteq 3,$$

$$\text{or } \frac{x}{2} - 3 \doteq \sqrt{(20x + 9)}.$$

Or let $y$ be taken for the number of sheep: then $(y - 9) \div 20$ is the number of scores.

$$\frac{y-9}{40} - \sqrt{y} \doteq 3.$$

$$\text{Hence } y - 40\sqrt{y} \doteq 129.$$

Completing the square, we find

$$y - 40\sqrt{y} + 400 = 529.$$

$$\therefore \sqrt{y} - 20 \doteq \pm 23.$$

*Exercise* 12.

Let $\frac{2}{3}x$, $\frac{3}{4}x$, and $\frac{4}{5}x$ be put for the three members.

$$\tfrac{4}{9}x^2 + \tfrac{16}{25}x^2 + 584 \doteq 2 \times \tfrac{9}{16}x^2 = \tfrac{9}{8}x^2.$$

Multiplying by 1800,

$$800x^2 + 1152x^2 + 1{,}051{,}200 \doteq 2025x^2.$$

$$\text{That is, } 73x^2 \doteq 1{,}051{,}200.$$

## Exercises in Chapter XVIII, Problem I.

(1.) Ans. 122. (2.) Ans. $-25a$.

## Exercises in Chapter XVIII, Problem II.

*Exercise* 1.

$$\text{Seventh term } \doteq 5a + 18b.$$

$$\text{Sum } \doteq \frac{10a + 18b}{2} \times 7 = 35a + 63b.$$

*Exercise* 2.

First term $\doteq 2$ yards.
Last term $\doteq 200$ yards.
Sum $\doteq 101 \times 100 = 10{,}100$.

## EXERCISES IN CHAPTER XIX, THEOREM.

*Exercise* 1.

Let $x$ be the tenth term. Then, by the theorem,

$$\frac{a}{b} \times x \doteq \frac{2}{3} \times 1\frac{1}{2} = 1.$$

$$\therefore x \doteq 1 \div \frac{a}{b} = \frac{b}{a}.$$

*Exercise* 2.

Let $y$ be the ninth term; and, by the theorem, we shall have

$$3x \times y = \left(\frac{4x^5}{3}\right)^2 = \frac{16}{9}x^{10}.$$

$$\therefore y \doteq \frac{16}{27}x^{10} \div 3x = \frac{16}{27}x^9.$$

*Exercise* 3.*

Any sum placed out at compound interest increases in a geometrical progression, for the principal with which every year commences increases in the same ratio as that of the preceding year. The given principal is the first term; the first year's amount is the second term; the second year's amount is the third term, and so on.

Now, at 4 per cent, the common ratio of the progression is 1·04, for £1 put out at 4 per cent. for 1 year becomes £1·04, and any other sum increases in the same ratio.

$\therefore$ Fifth year's amount $\doteq$ sixth term $= 200 \times (1{\cdot}04)^5 = 200 \times 1{\cdot}21665 + = 243{\cdot}33 + =$ £243 : 6 : 7+, Ans.

## EXERCISES IN CHAPTER XIX, PROBLEM I.

*Exercise* 1.

Tenth term $\doteq ({\cdot}0016) \times (5a)^9 = 3125a^9.$

*Exercise* 2.

Ninth term $\doteq \left(\frac{a}{2}\right)^4 \times \left(\frac{2}{a}\right)^8 = \left(\frac{2}{a}\right)^4.$

* This should have been placed under Problem I.

### EXERCISES IN CHAPTER XIX, PROBLEM II.

*Exercise* 1.

Last term $= xy^6 \doteq 256 \times (\frac{3}{2})^6 = 256 \times \frac{729}{64} = 2916.$
$(2916 - 256) \div \frac{1}{2} \doteq 2660 \times 2 = 5320.$
Sum $\doteq 5320 + 2916 = 8236.$

*Exercise* 2.

Last term $\doteq 1 \times 2^{20} = 1{,}048{,}576$ farthings.

$$\text{Sum} \doteq \frac{1{,}048{,}575}{1} + 1{,}048{,}576 = 2{,}097{,}151 \text{ far.}$$
$$= £2184 : 10 : 7\tfrac{3}{4}.$$

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# CHAPTER XXI.

## USEFUL FORMULÆ.

NOTE. This chapter has no counterpart in the "Course"; but it may be serviceable to the Teacher, as it contains a synopsis of the formulæ which are of most frequent use in elementary Algebra.

$$(a \pm b)^2 \doteq a^2 \pm 2ab + b^2.$$
$$(a \pm b)^3 \doteq a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$
$$(a \pm b)^4 \doteq a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4.$$
$$(a \pm b)^5 \doteq a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5.$$
$$(a \pm b)^6 \doteq a^6 \pm 6a^5b + 15a^4b^2 \pm 20a^3b^3 + 15a^2b^4 \pm 6ab^5 + b^6.$$
$$(a \pm b)^n \doteq a^n \pm na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \text{\&c.}$$

$$(a + b) + (a - b) \doteq 2a.$$
$$(a + b) - (a - b) \doteq 2b.$$
$$(a + b)^2 + (a - b)^2 \doteq 2(a^2 + b^2).$$
$$(a + b)^2 - (a - b)^2 \doteq 4ab.$$

$$(a+b)^3+(a-b)^3 \doteq 2(a^3+3ab^2).$$
$$(a+b)^3-(a-b)^3 \doteq 2(3a^2b+b^3).$$
$$(a+b)^4+(a-b)^4 \doteq 2(a^4+6a^2b^2+b^4).$$
$$(a+b)^4-(a-b)^4 \doteq 2(4a^3b+4ab^3).$$

$$(a-b)\times(a+b) \doteq a^2-b^2.$$
$$(a-b)\times(a^2+ab+b^2) \doteq a^3-b^3.$$
$$(a-b)\times(a^3+a^2b+ab^2+b^3) \doteq a^4-b^4.$$
$$(a-b)\times(a^4+a^3b+a^2b^2+ab^3+b^4) \doteq a^5-b^5.$$
$$(a-b)\times(a^n+a^{n-1}b+a^{n-2}b^2\ldots\ldots\ldots+b^n) \doteq a^{n+1}-b^{n+1}.$$

$$(a+b)\times(a-b) \doteq a^2-b^2.$$
$$(a+b)\times(a^2-ab+b^2) \doteq a^3+b^3.$$
$$(a+b)\times(a^3-a^2b+ab^2-b^3) \doteq a^4-b^4.$$
$$(a+b)\times(a^4-a^3b+a^2b^2-ab^3+b^4) \doteq a^5+b^5.$$
$$(a+b)\times(a^n-a^{n-1}b+a^{n-2}b^2\ldots\ldots\ldots\pm b^n) \doteq a^{n+1}\pm b^{n+1}.$$

$$(a+b)^2\times(a-b) \doteq a^3+a^2b-ab^2-b^3.$$
$$(a-b)^2\times(a+b) \doteq a^3-a^2b-ab^2+b^3.$$
$$(a+b)^2\times(a-b)^2 \doteq a^4-2a^2b^2+b^4.$$

END OF PART FIRST.

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## ERRATA.

| | | |
|---|---|---|
| Page | 8. | Line 4. For $-10\sqrt{bc}$ read $-20\sqrt{bc}$. |
| ... | 8. | Ex. 5. Ans. For $-8c^2$ read $+8c^2$. |
| ... | 10. | Ex. 13. Ans. For $-51$ read $+51$. |
| ... | 20. | Ex. 2. For $+30a$ read $-30a$. |
| ... | 20. | Note 1, line 2. For *sign* read *signs*. |
| ... | 23. | Last line. For $x-y$ read $x-2$. |
| ... | 33. | Ex. 15. Ans. For $18a$ read $18ax$. |
| ... | 53. | Ex. 6. For $y=40$ read $y=10$. |
| ... | 68. | Ex. 3. Ans. For $y$ read $z$. |
| ... | 71. | Ex. 12. Ans. For 48 read 40. |
| ... | 82. | Ex. 6. For $x$ read $a$ throughout. |
| ... | 85. | Line 3. For *fifth* read *third*. |
| ... | 85. | Ex. 2, in Ch. XI, Pr. II. For 96 read 192. |
| ... | 85. | Ex. 4, ...................... For $6a^3b^3$ read $12a^3b^3$. |
| ... | 95. | Line 1. For 39 read 49. |